Calculus of Variations
and
Optimal Control

August 13, 2014

The course material will be presented by the student(s) in the form of lectures and discussions at the following 11 sessions, covering material from: Liberzon, D. (2012). *Calculus of variations and optimal control theory: a concise introduction.* Princeton University Press.

Session 1: The basic calculus of variations problem

Section 1.3: *Preview of infinite-dimensional optimization*

Norms on function spaces, functional derivatives, and necessary conditions for optimality.

Section 2.2: *Basic calculus of variations problem*

Formulation of the standard problem and definition of strong and weak extrema.

Session 2: Euler-Lagrange equations

Section 2.3.1: *Euler-Lagrange equations*

Derivation of the EL-eqs.

Section 2.3.3-2.3.5: *Technical remarks and special cases*

Differentiability assumptions for the EL-eqs., special Lagrangians, and variable endpoint problems.

Session 3: Hamiltons equations

Section 2.4.1: *Hamiltons canonical equations*

Derivation of the canonical equations from the EL-eqs.

Section 2.4.1: *The Legendre transformation*

Relation between the Lagrangian and the Hamiltonian.

Session 4: Constraints and Second-order conditions

Section 2.5: *Variational problems with constraints*

Integral and non-integral constraints.

Section 2.6: *Second-order conditions*

Necessary and sufficient conditions for weak extrema.

Session 5: Motivation for optimal control

Section 3.1: *Necessary conditions for strong extrema*

Weierstrass-Erdmann corner conditions and Weierstrass excess function. *Only present on a conceptual level; no details.*

Section 3.2–3.3: *Optimal control problem formulation and assumptions*

Difference between calculus of variations and optimal control, optimal control problem formulation and assumptions,

Session 6: Statement of the Maximum Principle (MP)

Section 3.4: *The fixed-time, free-endpoint problem*

The variational approach and its shortcomings.

Section 4.1: *The MP*

Statement of the maximum principle for the basic fixed and variable endpoint problems.

Session 7: Proof of the MP

Section 4.2.0: *Proof structure*

Overview of the structure of the proof of the MP.

Section 4.2.1: *Equivalence of problem formulations*

Constructive proof that for each problem in the Lagrange form there is an equivalent problem in the Mayer form.

Section 4.2.2-4.2.5: *Perturbations and their propagation.*

Construction of a temporal perturbation, a spatial perturbation and its propagation in time, and their effect on the final state.

Session 8: Proof of the MP (cont.)

Section 4.2.6-4.2.7: *Some topology*

Construction of the adjoint vector as the normal of a hyperplane separating the cone of perturbation directions from the direction of decreasing cost.

Section 4.2.8: *Propagation of the adjoint vector*

Derivation of the adjoint equation.

Section 4.2.9: *The Hamiltonian maximization condition*

Proof that the Hamiltonian is maximized for optimal controls, and is constant in time along the optimal path.

Session 9: Discussion and application of the MP

Section 4.2.10: *The transversality condition*

The topological statements are refined to obtain the transversality condition needed for variable endpoint problems.

Section 4.3: *Technicalities and changes of variables*

Fixed terminal time, time dependent system and cost, terminal cost, initial sets.

Session 10: The Hamilton-Jacobi-Bellman equation

Section 5.1: *Derivation of the HJB equation*

Discrete dynamic programming, the value function and the principle of optimality, examples, and sufficient optimality conditions.

Session 11: Discussion of the HJB equation

Section 5.2: *Comparison of the HJB equation and the MP*

The HJB PDE vs. the MP ODEs. Proving the MP using the HJB equation? Differentiability assumptions.

Section 5.3: *Viscosity solutions of the HJB equation*

Generalized solutions to PDEs in general and the HJB equation in particular.