Communications and Transport Systems
Department of Science and Technology **Fall 2019**
Linköping University

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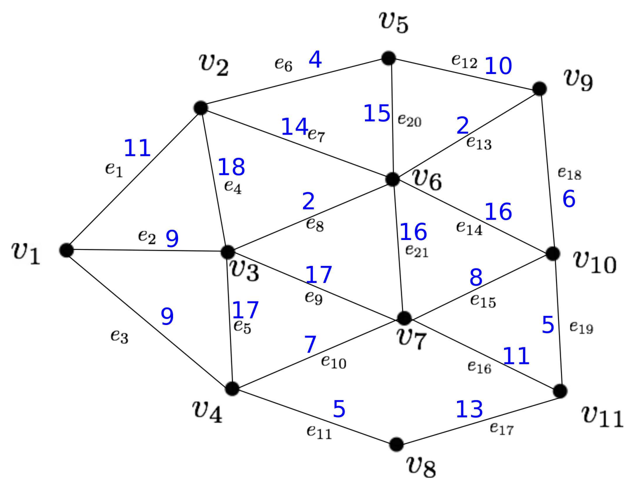
Design and Analysis of Algorithms Part 1 - Mathematical tools and Network problems homework 3, 18.11.2019

Problem 1 (Prim's algorithm):



Use Prim's algorithm to determine a minimum spanning tree; start with vertex v_1. (Note: we break ties, when several vertices could be chosen in an iteration, by choosing the vertex with the lowest index.)

Problem 2 (Independence Systems and Matroids):
Given an undirected, connected graph G. Let $E = E(G)$,
$\mathcal{I} = \{F \subseteq E : F$ is subset of an Hamiltonian circuit in $G\}$.

(a) Show that the set system $(E, \mathcal{I})$ is an independence system.

(b) Test whether the set system $(E, \mathcal{I})$ is a matroid.

Let E_2 be a finite set, k a positive integer, and $\mathcal{I}_2 = \{F \subseteq E_2 : |F| \leq k\}$.

(c) Test whether the set system $(E_2, \mathcal{I}_2)$ is a matroid.

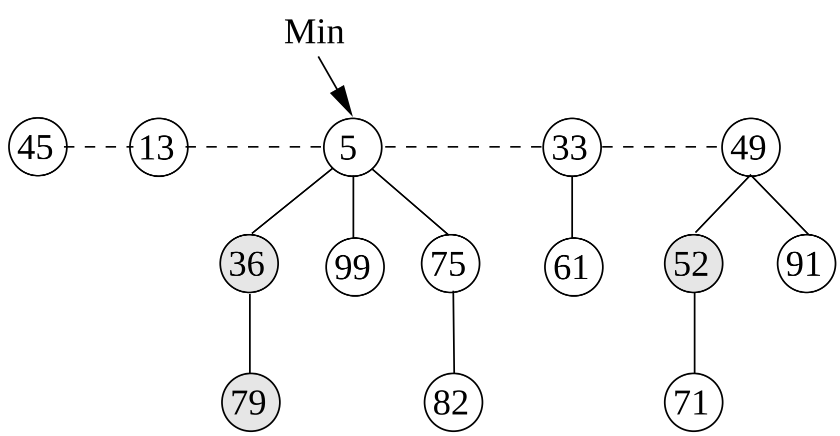
Problem 3 (Trees): Let (V, T_1) and (V, T_2) be two trees on the same set of vertices V. Show: For each edge $e \in T_1$ there exists an edge $f \in T_2$, such that both $(V, (T_1 \setminus \{e\}) \cup \{f\})$ and $(V, (T_2 \setminus \{f\}) \cup \{e\})$ are trees.

Problem 4 (Minimum Spanning Trees):
Given an undirected, connected graph $G = (V, E)$. Prove or disprove the following statement:
If G contains a cycle with a unique lightest edge e (= edge of lowest weight), e is contained in every MST.

Problem 5 (Bottleneck Spanning Trees):
Give a linear time algorithm that for a given graph G and an integer b computes whether the value of a bottleneck spanning tree is at most b.

Problem 6 (Fibonacci Heaps):

Fibonacci heaps allow for an efficient implementation of Dijkstra's shortest paths algorithm. More detailed information on this data structure can, for example, be found here `https://www.cs.princeton.edu/~wayne/teaching/fibonacci-heap.pdf`, or in Chapter 19 of the book Introduction to Algorithms by Cormen, Leiserson, Rivest, and Stein.
In the following Fibonacci heap marked vertices are shown in gray.



Execute the following operations one after another on the given Fibonacci heap:

INSERT(42), DELETE_MIN, DECREASE_KEY(91 to 60), DECREASE_KEY(71 to 61)

Depict each resulting Fibonacci heap (possibly with intermediate states).

Problem 7 (Spanning tree):
The *tree graph* $T(G)$ of a connected graph G has a vertex for every spanning tree of G. Two of these tree vertices are adjacent if they have $|V| - 2$ edges in common. Show that $T(G)$ is connected.

Problem 8 (MSTs):
Let (G, w) be a network and v an arbitrary vertex. Show that each MST must contain an edge incident to v with the smallest weight of all edges incident to v.

Problem 9 (Independent sets):
An *independent set* (IS) or *stable set* is a set of vertices in a graph, no two of which are adjacent. An independent set that is not the subset of another independent set is called maximal. A maximal IS is a *dominating set*, that is,a subset D of V such that every vertex not in D is adjacent to at least one member of D.
Algorithm 1 computes a maximal independent set.

a) What is its runtime?

b) Show that Algorithm 1 does not compute a maximum independent set.

c) Show: If G can be vertex colored with k colors, there exists a vertex u with degree at most $\lfloor (1 - \frac{1}{k})|V| \rfloor$. (Hint: We do not know k, we only use that k is the optimal number of colors and that $k \geq 2$.)

d) Show: If G can be vertex colored with k colors, the size of the independent set found by Algorithm 1 is at least $\lceil log_k(|V|/3) \rceil$.

Algorithm 1: Greedy IS
Input : Undirected graph $G = (V, E)$ with at least two vertices.
Output: A maximal independent set in G.
$U = \varnothing$
1.WHILE the graph is not empty
  Choose some vertex u with minimum degree.
  Remove u and all its neighbors from G.
  $U = U \cup u$.
RETURN U.

Problem 10 (Vertex coloring algorithm):
For any color, the vertices with this color form an independent set. Recall that we can find a maximal independent set in polynomial time
Consider Algorithm 2. Assume G can be colored with k colors.

a) Show that after at most $\frac{n}{1/2log_k(n/16)}$ steps (maybe less!), at most $\frac{n}{log_k(n/16)}$ uncolored vertices remain.

b) Algorithm 2 uses at most $\frac{3n}{log_k(n/16)}$ colors.

c) How far off is Algorithm 2 from the optimum of k?

Algorithm 2: Greedy Vertex Coloring
Input : Undirected graph $G = (V, E)$.
Output: A feasible vertex coloring of G.
1.WHILE the graph is not empty
    Determine a large independent set U using Algorithm 1.
    Color all vertices in U with the same color.
    Remove U from G.

Problem 11 (Vertex coloring in planar graphs):
Let $G = (V, E)$ be a planar graph. Show: If G has no cycle of odd length, it is 2-colorable.

Problem 12 (Vertex coloring algorithm for planar graphs):
Consider Algorithm 3. We have to figure out that a vertex u as chosen by the algorithm, that is, a vertex of degree at most five, always exists.

a) Show: Let G be a connected planar graph, and let n, m and f denote the numbers of vertices, edges, and faces, respectively, in a plane drawing of G. Then $n - m + f = 2$.

b) Show: $m \leq 3n - 6$

c) Show: a vertex u as chosen by the algorithm always exists, that is, there is a vertex with degree at most 5.

d) How many colors does Algorithm 3 use at most?

Problem 13 (Independence Systems): Let $E = \{1, \dots, 10\}$ and
$\mathcal{I}_1 = \{\{1,2,3,4\}, \{2,3,4,5\}, \{3,4,5,6\}, \{4,5,6,7\}, \{5,6,7,8\}, \{6,7,8,9\},$
$\{7,8,9,10\}, \{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,6\}, \{5,6,7\}, \{6,7,8,\},$
$\{7,8,9\}, \{8,9,10\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,7\}, \{7,8\},$
$\{8,9\}, \{9,10\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \varnothing\}$
and
$\mathcal{I}_2 = \{\{1,2,3\}, \{6,7,9\}, \{1,2\}, \{1,3\}, \{2,3\}, \{6,7\}, \{6,9\}, \{7,9\}, \{1\}, \{2\}, \{3\}, \{6\}, \{7\}, \{9\}\}$

a) Is $(E, \mathcal{I}_1)$ an independence system?

Algorithm 3: Vertex Coloring for Planar graphs
Input : Undirected planar graph $G = (V, E)$.
Output: A feasible vertex coloring of G.
1. IF two colors are sufficient (problem 8) DO color G using two colors.
2. ELSE
Find an uncolored vertex u with degree at most 5.
Remove u and all its adjacent edges and color the remaining graph recursively.
Insert u and its adjacent edges back and color u with a color that none of its neighbors has

b) Is $(E, \mathcal{I}_2)$ an independence system?

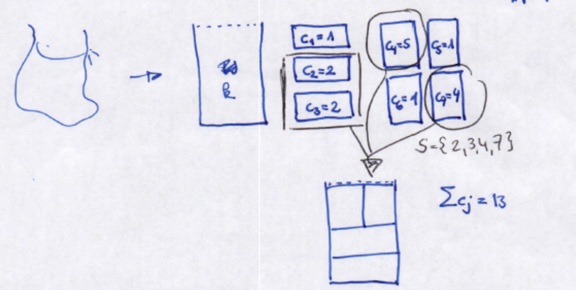
Problem 14 (Independence Systems II):

The Knapsack Problem

Given: Nonnegative integers $n, c_1, \dots, c_n, w_1, \dots, w_n$ and k

Task: Find a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{j \in S} w_j \le k$ and $\sum_{j \in S} c_j$ is maximum

w – „weight"
c – „profit"



Let's reformulate the Problem:

KNAPSACK PROBLEM

Given nonnegative numbers c_i, w_i $(1 \le i \le n)$, and k, find a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{j \in S} w_j \le k$ and $\sum_{j \in S} c_j$ is maximum.

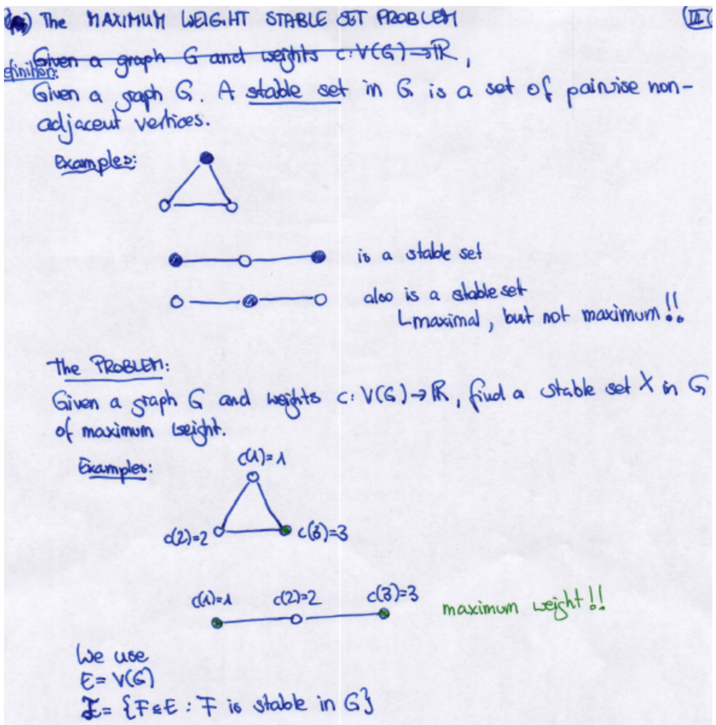
We use

$E = \{1, \dots, n\}$

$\mathcal{I} = \{F \subseteq E : \sum_{j \in F} w_j \le k\}$

Is $(E, \mathcal{I})$ an independence system?

Problem 15 (Independence Systems II):



Is $(E, \mathcal{I})$ an independence system?

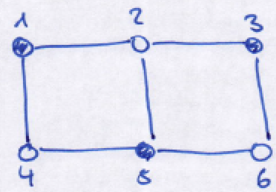
Problem 16 (Matroids): Let $E_2 = \{1, \ldots, 7\}$ and
$\mathcal{I}_3 = \{\{1,2,3\}, \{1,2,5\}, \{1,2,6\}, \{1,2,7\}, \{1,3,4\}, \{1,3,6\}, \{1,3,7\}, \{1,4,5\}, \{1,4,6\},$
$\{1,4,7\}, \{1,5,6\}, \{1,5,7\}, \{2,3,4\}, \{2,3,5\}, \{2,3,7\}, \{2,4,5\}, \{2,4,6\}, \{2,4,7\}, \{2,5,6\},$
$\{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{3,5,7\}, \{4,5,7\}, \{5,6,7\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\},$
$\{1,7\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{2,7\}, \{3,4\}, \{3,5\}, \{3,6\}, \{3,7\}, \{4,5\}, \{4,6\}, \{4,7\},$
$\{5,6\}, \{5,7\}, \{6,7\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \varnothing\}$

a) Is $(E_2, \mathcal{I}_3)$ an independence system?

b) Is $(E_2, \mathcal{I}_3)$ a matroid?

Let E be the set of edges of some undirected graph G, S a stable set in G, k_s integers ($s \in S$) and
$\mathcal{I} := \{F \subseteq E : |\delta_F(s)| \le k_s \ \forall s \in S\}$

Example:

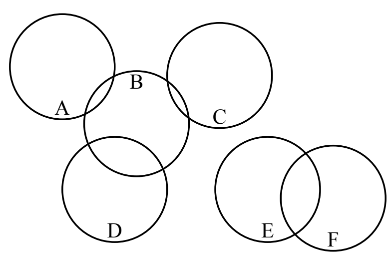


$S = \{1,3,5\}$
$k_1 = 2$
$k_3 = 1$
$k_5 = 3$

$\Rightarrow I = \{\{1,2\},\{2,3\},\{1,4\}\} \in \mathcal{I}$
but
$I = E \notin \mathcal{I}$, as $|\delta_E(3)| = \overset{2}{\cancel{3}} \not\le 1$

Problem 17 (Matroids II):
Is $(E, \mathcal{I})$ a matroid?

Problem 18 (IS and matroids): Consider the following system: We are given a ground set, consisting of circles with uniform radius in the plane. For an example:



We say that a selection of some of these circles is *independent*, iff no two of them intersect. For example, $\{C, D\}$ is independent, but $\{E, F\}$ is dependent.

a) Prove that this system is an independence system for any given ground set of circles.

b) Find a nonempty example of circles for which the system is a matroid (and prove it).

c) Find an example of circles for which the system fulfills these criteria:

- It is **not** a matroid (prove it).
- All bases have the same size k, with $k \geq 3$.