Matroid Theory & Greedy Algorithm

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Outline

Independence Systems: Motivation

- A set of items
- You can choose an arbitrary number of those items (i.e. a subset)
- Some property that holds when no items have been chosen (i.e. the empty subset)
- As more items are chosen, that property can become broken, but can never become unbroken again

Independence Systems: Examples

Some examples of Independence systems are-

- A graph G and the family of ...
  - ...all planar subgraphs of G
  - ...all matchings for G
  - ...all cliques on G
- A set of objects S and the family of all subsets of S that fit into some given knapsack

Independence Systems: Definition

Definition

A set system $(E, \Im)$, where E is a finite set and $\Im$ is a set of subsets of E, is an independence system if
(IS.1) $\emptyset \in \Im$
(IS.2) $I \in \Im,\ J \subset I \implies J \in \Im$

The elements of $\Im$ are called **independent**, the elements of $2^E \backslash \Im$ are called **dependent**.

More on Independence systems

Theorem

Let $(E,\Im)$ be an independence system, then the following statements are equivalent-

1. *If* $X, Y \in \Im$ *and* $|X| > |Y|$ *then there is an* $x \in X \setminus Y$ *with* $Y \cup \{x\} \in \Im$.
2. *If* $X, Y \in \Im$ *and* $|X| = |Y| + 1$, *then there is an* $x \in X \setminus Y$ *with* $Y \cup \{x\} \in \Im$.
3. *For each* $X \subseteq E$, *all bases of* X *has the same cardinality.*

Matroids

Definition

An independence system is a **matroid** if $X, Y \in \Im$ and $|X| > |Y|$, then there is an $x \in X \backslash Y$ with $Y \cup \{x\} \in \Im$

This property is known as the augmentation or exchange property

Matroids: Examples

Some examples of matroids are-

- A graph G and the family of all subgraphs of G that are forests (a.k.a. a graphic matroid).
- A set of vectors V and the family of subsets of V in which all vectors are linearly independent.
- All matroids are independence system but not all independence systems are matroids. For example matchings in a graph.

Bases, circuits and rank

Definition

A base is a maximal independent set.

Definition

A circuit is a minimal dependent (i.e. not independent) set.

Definition

Let $(E, \Im)$ be an independence system. For $X \subseteq E$, we define the rank of X by $r(X) := max\{|Y| : Y \subseteq X, Y \in \Im\}$.

The rank of a matroid is the maximum size of an independent set in the matroid.
All bases have the same cardinality (proof!).
A circuits don't necessarily have the same cardinality (give example).

Bases and related theorem

Theorem

Let E be a finite set and $\mathcal{B} \subseteq 2^E$. $\mathcal{B}$ is the set of bases of some matroid $(E, \Im)$ if and only if the following holds:
(B1) $\mathcal{B} \neq \emptyset$;
(B2) For any $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \backslash B_2$, there exists a $y \in B_2 \backslash B_1$ with $(B_1 \backslash \{x\}) \cup \{y\} \in \mathcal{B}$

Duality of Independence systems

Definition

Let $(E, \Im)$ be an independence systems. We define the **dual** as $(E, \Im^*)$, where
$\Im^* = \{I \subseteq E\colon$ there is a basis B of $(E, \Im)$, such that $I \cap B = \emptyset\ \}$

Dual of an independence system is also an independence system.

Matroid Duality

Proposition

$(E, \Im^{**}) = (E, \Im)$.

Proof.

$I \in \Im^{**} \iff$ there is a basis B^* of $(E, \Im)$ such that $F \cap B^* = \emptyset \iff$ there is a basis B of $(E, \Im)$ such that $I \cap (E \backslash B) = \emptyset \iff I \in \Im$ □

Theorem

Let $(E, \Im)$ be an independence system, $(E, \Im^*)$ its dual, and let r and r^* be the corresponding rank functions.

1. *$(E, \Im)$ is a matroid iff $(E, \Im^*)$ is a matroid.*
2. *If $(E, \Im)$ is a matroid then, $r^*(I) = |F| + r(E \backslash F) - r(E)$ for $F \subseteq E$*

Matroid Duality: proof of the first theorem

Definition

Let $M(IS) = \begin{cases} \text{true}, & IS \text{ is a matroid} \\ \text{false}, & \text{otherwise} \end{cases}$

Proof.

Assume that $M((E, \Im))$ and $\neg M((E, \Im^*))$. Hence, $\exists\, B_1^*, B_2^*$ bases of $(E, \Im^*)$ such that $|B_1^*| > |B_2^*|$. Since $(E, \Im^*)$ is a dual of $(E, \Im)$, $\exists\, B_1, B_2$ bases of $(E, \Im)$ such that $B_1 \cap B_1^* = \emptyset, B_2 \cap B_2^* = \emptyset$. $M((E, \Im))$, so $|B_1| = |B_2|$, and $|B_2| + |B_2^*| < |B_1| + |B_1^*| \leq |E|$, so $\exists\, x \in E \backslash (B_2 \cup B_2^*)$. Now, $(B_2^* \cup \{x\}) \cap B_2 = (B_2^* \cap B_2) \cup (\{x\} \cap B_2) = \emptyset \cup \emptyset = \emptyset$, so $(B_2^* \cup \{x\}) \in \Im^*$, hence B_2^* is not a maximal independent set in the dual, which contradicts the fact that B_2^* is a base, so $M((E, \Im)) \Rightarrow M((E, \Im^*))$. We also know that $(E, \Im^{**}) = (E, \Im)$, so $M((E, \Im^*)) \Rightarrow M((E, \Im^{**})) \Leftrightarrow M((E, \Im))$. Together, $M((E, \Im)) \iff M((E, \Im^*))$, Q.E.D. □

Weighted independence systems and matroids

- A weighted independence system is an independence system $IS = (E, \mathcal{I})$ for which each edge $e \in E$ is associated with a weight through a weighting function $w : E \to \mathbb{R}^+$.
- The weight of an subset of E is the total weight of the elements of the subset:

$$w(S) = \sum_{x \in S} w(x),\ S \in E$$

- A maximum-weight independent set is an independent set I with maximum total weight:

$$I \in \mathcal{I},\ w(I) = \max_{I' \in \mathcal{I}} w(I')$$

The Greedy Algorithm

INPUT: Weighted IS $(E, \mathcal{I})$ with weights $w(e) \geq 0 \ \forall e \in E$
OUTPUT: Independent set

1. Sort E in descending weight order, such that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_n)$
2. $B := \emptyset$
3. For $i = 1, \ldots, m$ DO:
   IF $B \cup \{e_i\} \in \mathcal{I}$:
   $B := B \cup \{e_i\}$
4. OUTPUT B

Greedy optimality

Theorem

6.6 Let $(E, \Im)$ be a matroid. Then:
(*)For each weight function with $w(e) \geq 0 \; \forall \, e \in E$, greedy finds a maximum-weight basis.

Greedy lemmas

Lemma

The independent set I given by greedy will be maximal

Proof.

Let $i_1, i_2, \dots, i_k$ be the indices of the elements in I and assume that I is not maximal. Then there is an element $e_{i'}$ with index i' such that $I \cup \{e_{i'}\} \in \mathcal{I}$. Denote by B_i the value the set B in greedy will have after iteration i, and note that $B_i \subseteq I$. Now, $B_{i'-1} \cup \{e_{i'}\} \subseteq I \cup \{e_{i'}\} \in \mathcal{I}$. However, at iteration i', $e_{i'}$ was not chosen by greedy since it is not in I, so $B_{i'-1} \cup \{e_{i'}\} \notin \mathcal{I}$, which is a contradiction. Hence, our assumption must be wrong and I must be maximal. □

Proof of Greedy optimality

Given, $(E, \Im)$ is a weighted matroid with weights w.

Let $e_1, e_2, \ldots, e_k$ be the elements chosen by the greedy algorithm and $f_1, \ldots, f_k$ be the elements of an arbitrary basis, and assume that they are sorted in descending order, i.e. $e_1 \geq e_2 \geq \cdots \geq e_k \geq 0$ and $f_1 \geq f_2 \geq \cdots \geq f_k \geq 0$