Pseudo-Marginal Metropolis Light Transport

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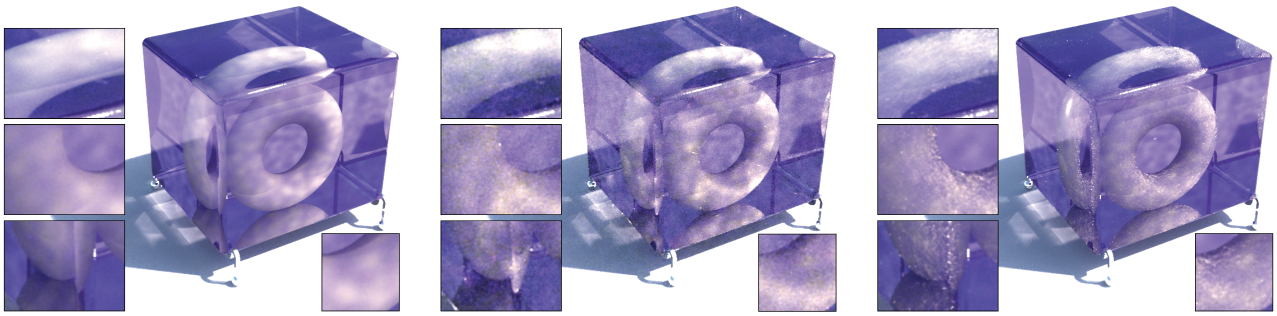


(a) Pseudo-marginal ERPT using ratio tracking (b) ERPT with ray marching (c) PSSMLT with ratio tracking

Figure 1: *Equal time renderings of a refractive glass cube containing isotropic heterogenous participating media and a diffuse torus. a) Our method using ERPT [Cline et al. 2005] with ratio tracking [Novák et al. 2014] to obtain an unbiased estimate of the transmission. b) Same as a) but with deterministic ray-marching used to evaluate the transmission. c) PSSMLT [Kelemen et al. 2002] with ratio tracking.*

1 Introduction

Accurate and efficient simulation of light transport in heterogeneous participating media, such as smoke, clouds and fire, plays a key role in the synthesis of visually interesting renderings for e.g. visual effects, computer games and product visualization. Similar problems also has important applications in other fields such as neutron transport and medical radiation dosimetry. While significant progress has been made towards robust and efficient algorithms for rendering heterogeneous participating media, scenes containing highly anisotropic phase functions, glossy surfaces and complex visibility still remain challenging.

Modern global illumination rendering is based on Monte Carlo methods computing averages of stochastically sampled light paths connecting the sensor with light sources in the scene. The efficiency of these algorithms is directly dependent on the strategy used in the light path sampling. For complex scenes, the Metropolis light transport (MLT), [Veach and Guibas 1997], algorithm provide a powerful framework for sampling the path space. The method is based on using Markov chain Monte Carlo (MCMC) to sample light paths $\{\bar{x}^i\}_{i\geq 1}$ according to some target distribution, $\pi(\bar{x})$, proportional to the image contribution of the paths.

However, in many applications the target distribution, $\pi(\bar{x})$, cannot be evaluated exactly. For example, when rendering scenes with heterogenous participating media, where, in general, computing the transmittance along a path is analytically intractable. This limits direct use of MLT, as it depends on Metropolis-Hastings (MH) acceptance probabilities which cannot be evaluated in closed form. The same problem occur for other rendering methods relying on MCMC sampling, such as Energy Redistribution path tracing (ERPT) [Cline et al. 2005]. Previous work, [Pauly et al. 2000], proposed to substitute the exact quantity of interest, $\pi(\bar{x})$, with a biased approximation, $\tilde{\pi}(\bar{x})$, to compute the MH probability. However, direct application of this method does not to lead to a consistent estimator.

Our main contribution is a new sampling strategy to be used within existing MCMC based rendering methods, e.g. MLT and ERPT, for rendering of scenes with heterogenous participating media. Specifically, we show that any positive and unbiased estimator of the target distribution $\mathrm{E}\left[\widehat{\pi}(\bar{x})\right] = \pi(\bar{x})$ can replace the exact quantity to simulate a Markov Chain with a stationary distribution that has a marginal that is the *exact* target distribution of interest. This enables us to evaluate the transmittance function with recent unbiased estimators, [Novák et al. 2014], leading to significantly shorter rendering times, see e.g. Figure 1. Our approach is based on pseudo-marginal MCMC methods developed for Bayesian inference with intractable likelihoods, [Beaumont 2003; Andrieu and Roberts 2009]. Compared to previous work (relying on *biased* ray-marching for evaluating transmittance), our method enables simulation of longer Markov chains, a better exploration of the path space, and consequently less image noise.

2 Background and related work

2.1 Path Integral Formulation

Using the path integral formulation of light transport [Veach 1997; Pauly et al. 2000; Wenzel 2013], the j-th pixel measurement I_j is expressed as an integral over the space, Ω, of all valid light transport paths of all lengths:

$$I_j = \int_{\Omega} f(\bar{x}) W_j d\mu(\bar{x}), \text{ with } d\mu(\bar{x}) = \prod_{i=0}^{k} \mathrm{d}\mu(x_i). \quad (1)$$

Here a valid path of length $k \in (0, \infty)$, $\bar{x} = x_0, ..., x_k$ connects a light source to the sensor through a series of scattering vertices, x_i, located on surfaces or in participating media. Here $W_j = W_j(x_{k-1} \to x_k)$ denotes the pixel sensitivity/filter at the sensor. The path measure, $d\mu(\bar{x})$, is a product measure over the measures of the path vertices, where path vertices on surfaces correspond to area integration, $d\mu(x_i) = dA(x_i)$, and vertices in a media correspond to volume integration, $d\mu(x_i) = dV(x_i)$. The measurement contribution function, $f_j(\bar{x})$, is defined by

$$f_j(\bar{x}) = L_e \left[\prod_{i=0}^{k-1} G(x_i, x_{i+1}) T(x_i, x_{i+1}) \right] \left[\prod_{i=1}^{k-1} \rho(x_i) \right]. \quad (2)$$

where $L_e = L_e(x_0 \to x_1)$ is the emitted radiance at the light source, $\rho(x_i)$ denotes scattering functions defined at path vertices, $G(x_i, x_{i+1})$ is the geometry term, and $T(x_i, x_{i+1})$ is the transmittance defined on the segments between path vertices. For a detailed description of these terms see e.g. [Wenzel 2013].

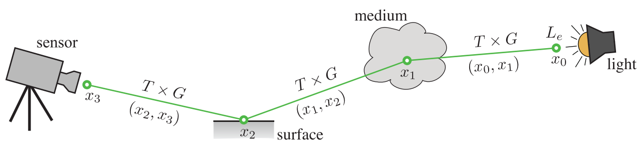


Figure 2: *Illustration of the notation for describing the light transport in the scene using the path integral formulation.*

2.2 Evaluating transmittance

The transmittance between two path vertices, x_i and x_{i+1}, separated by a distance d is given by:

$$T(x_i, x_{i+1}) = \exp\Big(- \int_0^d \sigma(x + t\omega_{i,i+1})\mathrm{d}t \Big) \qquad (3)$$

where $\sigma(x)$ denotes the extinction coefficient describing the loss of light due to absorption and out-scattering per unit distance, and $\omega_{i,i+1}$ denotes a unit length vector pointing from x_i towards x_{i+1}. A challenge is that the transmittance only can be evaluated analytically in special cases, and is intractable for general heterogenous media. Ray-marching methods [Perlin and Hoffert 1989] evaluate the inner integral $\int_0^d \sigma(x)\mathrm{d}t$ using standard quadrature methods. However, these techniques are often computationally inefficient and the resulting transmission estimates are biased. Instead a algorithm related to rejection sampling, known as *delta tracking*, Woodcock tracking or pseudo tracking can be used. This method was developed independently in neutron transport [Woodcock et al. 1965] and plasma physics [Skullerud 1968] for sampling free-flight distances in heterogenous media given an upper bound of the extinction coefficient, $\bar{\sigma}(x) \geq \sigma(x), \forall x$. A detailed study of delta tracking was presented by Coleman [1968]. To evaluate the transmittance, a standard technique is to simply average M delta tracking samples and evaluate the ratio of samples corresponding to a distance longer than d. Improved transmittance estimators based on de-randomization and control variate techniques have also been proposed in the literature. In this paper we use the *ratio tracking* estimator proposed by Novak *et al.* [2014] providing a noisy but unbiased estimate of the transmittance function.

2.3 Metropolis light transport

To evaluate the path integral (1) the MLT method, [Veach and Guibas 1997], uses MCMC to sample paths from a target probability distribution, $\bar{x} \sim \pi(\bar{x}) = \frac{L(\bar{x})}{\int_\Omega L(\bar{x})\mathrm{d}\mu(\bar{x})}$, where $L(\bar{x})$ is a scalar contribution function. This function is usually constructed to be similar to the luminance of the path contribution function $f(\bar{x})$. The path integral is then estimated by the average, $\hat{I}_j = \frac{Z}{N}\sum_{i=1}^{N} \frac{W_j f(\bar{x}^i)}{L(x)}$, where $Z = \int_\Omega L(\bar{x})\mathrm{d}\mu(\bar{x})$ is an unknown normalizing constant that can be computed using traditional path tracing techniques. To sample paths from the target distribution, $\pi(\bar{x})$, a set of seed paths are first generated using traditional path tracing techniques. These initial seed paths are then iteratively updated with the MH algorithm to construct a Markov chain on Ω that is ergodic and has $\pi(\bar{x})$ as its stationary distribution. Given a current path, $\bar{x}^i$, a new path, $\bar{x}^{i+1}$, is first proposed using a conditional proposal density, $q(\bar{x}^{i+1}|\bar{x}^i)$, and then accepted with probability

$$\min\left\{1, \frac{L(\bar{x}^{i+1})}{L(\bar{x}^i)} \frac{q(\bar{x}^i|\bar{x}^{i+1})}{q(\bar{x}^{i+1}|\bar{x}^i)}\right\} \qquad (4)$$

A sufficient condition for the resulting Markov chain to be irreducible is that q is positive everywhere. In practice, this is often met by mixing independent proposals with local exploration of the state space. The ERPT, [Cline et al. 2005], method is an extension to the original MLT algorithm, which first samples a large set of paths with standard path tracing methods, and then runs many short MCMC chains starting form each initial sample. Similarly to the original MLT algorithm, MH updates are used to simulate the Markov chains. The main benefit of EPRT is that a well stratified initial sample set can be preserved, e.g. using one path per pixel.

An extended formulation of MLT for rendering scenes with participating media was presented by Pauly *et al.* [2000]. For heterogenous media they used a ray-marching algorithm to evaluate the transmittance along sampled light paths, which often requires extensive computational effort and produces biased estimates. A problem (see discussion in the next section) is that if a biased estimate of the transmission is used in place of the exact quantity, the resulting Markov chain does not necessarily target the desired density $\pi(\bar{x})$. Kelemen *et al.* [2002] presented an alternative MCMC method. Instead of sampling directly in path space they propose to apply mutations to random number vectors in *primary sample space*, which are then used to generate light paths via traditional path tracing methods. We refer to this method as Primary sample space Metropolis light transport (PSSMLT). While the method is easy to implement it is limited to the underlying path tracing techniques restricting the possible exploration of the path space. Raab *et al.* [2008] introduced delta tracking to computer graphics and argued that it could be used to derive an unbiased path tracing algorithm that could then be used with PSSMLT. In this paper we significantly extend their work by showing that we can construct valid algorithms for the larger family of path space MLT and ERPT methods by employing unbiased estimators of the scalar contribution function.

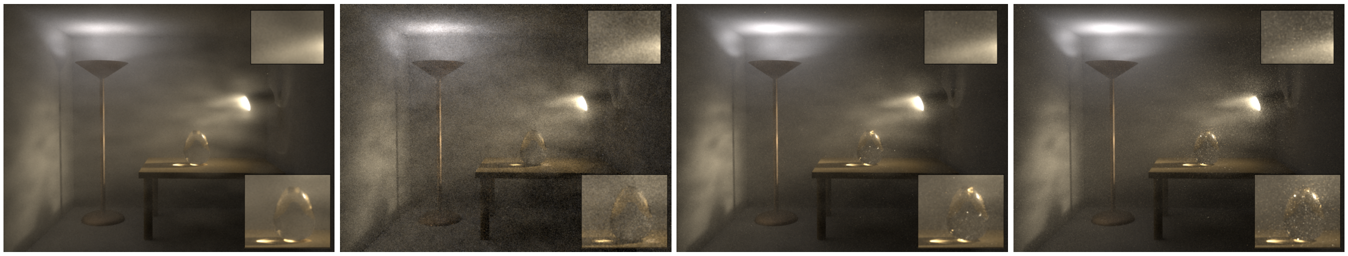
3 Replacing the contribution function with an unbiased estimate

In cases where the contribution function is intractable, e.g. rendering heterogenous media, the standard MH acceptance probability (4) cannot be evaluated. Instead, we propose to employ a pseudo-marginal construction [Andrieu and Roberts 2009], where a non-negative and unbiased estimator of the contribution function $\hat{L}(\bar{x})$ is used in place of its intractable counterpart $L(\bar{x})$. We show that this allows us to generate a Markov chain with the *exact* target distribution $\frac{L(\bar{x})}{Z}$ as its stationary distribution although we only have access to an *approximation* of the contribution function. By employing this construction inside previous MH based rendering algorithms, e.g. MLT and ERPT, we can not only increase their flexibility and applicability but also their efficiency.

Let us first introduce an auxiliary random variable u denoting all the random quantities generated to construct the estimator $\widehat{L}(\bar{x})$. This random variable follows a probability distribution $g(u|\bar{x})$ defined on some space $\mathcal{U}$. Consider now the joint distribution of the light path $\bar{x}$ and the auxiliary variable u

$$p(\bar{x}, u) = \frac{L(\bar{x})}{Z} g(u|\bar{x}). \qquad (5)$$

This distribution has the target distribution as one of its marginals, as $\int \frac{L(\bar{x})}{Z} g(u|\bar{x})\mathrm{d}u = \frac{L(\bar{x})}{Z}$. Inspired by this joint distribution we



(a) Pseudo-marginal MLT (b) MLT with ray marching (c) PSSMLT with ratio tracking (d) Bidirectional path tracing

Figure 3: *Equal time renderings of the classical scene from Veach filled with anisotropic scattering heterogenous media ($g = 0.85$).*

can construct an *extended target* distribution

$$\phi(\bar{x}, u) = \frac{\widehat{L}_u(\bar{x})}{Z} g(u|\bar{x}) \tag{6}$$

where we have replaced the exact contribution function with an unbiased estimator, $\widehat{L}_u(\bar{x})$. This extended target distribution can now be targeted directly with a standard MH algorithm in the extended space $\Omega \otimes \mathcal{U}$. The resulting algorithm will then in fact generate samples $\{\bar{x}^i\}_{i\geq 1}$ from the target distribution $\frac{L(\bar{x})}{Z}$ even though we substituted the exact contribution function with an approximation in (6). To see why this holds, let us consider the marginal with respect to u of the extend target distribution

$$\int \phi(\bar{x}, u) \mathrm{d}u = \frac{1}{Z} \int \widehat{L}_u(\bar{x}) g(u|\bar{x}) \mathrm{d}u. \tag{7}$$

As the contribution function estimate, $\widehat{L}_u(\bar{x})$, is unbiased we have

$$\mathrm{E}_{g(u|\bar{x})} \left[\widehat{L}_u(\bar{x}) \right] = \int \widehat{L}_u(\bar{x}) g(u|\bar{x}) \mathrm{d}u = L(\bar{x}) \tag{8}$$

which inserted into (7) proves that $\int \phi(\bar{x}, u)\mathrm{d}u = \frac{L(\bar{x})}{Z}$ is recovered *exactly* as the marginal of the extended target distribution (6) despite the fact that we used an *approximation* of the contribution function. An interpretation of these, so called, *exact approximation* algorithms is that using $\widehat{L}_u(\bar{x})$ does change the marginal with respect to u but it does not change the marginal with respect to $\bar{x}$.

To use a standard MH algorithm in the extended space, a new light path $\bar{x}^{i+1}$ and auxiliary variable u^{i+1} is proposed according to

$$\bar{x}^{i+1} \sim q(\bar{x}^{i+1}|\bar{x}^i), \quad u^{i+1} \sim g(u^{i+1}|\bar{x}^{i+1}), \tag{9}$$

based on the current sample pair $[\bar{x}^i, u^i]$. Note that, the simulation of u^{i+1} simply corresponds to generating a new estimate of the contribution function independently of previous estimates. The probability of accepting the new sample pair is then given by

$$\min \left\{ 1, \frac{\frac{1}{Z} \widehat{L}_{u^{i+1}}(\bar{x}^{i+1}) g(u^{i+1}|\bar{x}^{i+1})}{\frac{1}{Z} \widehat{L}_{u^i}(\bar{x}^i) g(u^i|\bar{x}^i)} \frac{q(\bar{x}^i|\bar{x}^{i+1}) g(u^i|\bar{x}^i)}{q(\bar{x}^{i+1}|\bar{x}^i) g(u^{i+1}|\bar{x}^{i+1})} \right\}$$

$$= \min \left\{ 1, \frac{\widehat{L}_{u^{i+1}}(\bar{x}^{i+1})}{\widehat{L}_{u^i}(\bar{x}^i)} \frac{q(\bar{x}^i|\bar{x}^{i+1}))}{q(\bar{x}^{i+1}|\bar{x}^i)} \right\} \tag{10}$$

As we are only interested in the sampled light paths $\{\bar{x}^i\}_{i\geq 1}$ and not the auxiliary quantities, we do not need to explicitly keep track of the generated auxiliary random variables $\{u^i\}_{i\geq 1}$. However, it is necessary to keep track of the estimated contribution of the light path $\widehat{L}_u(\bar{x})$, and keep this value for all comparisons with other proposed paths. Note that, a key requirement for the marginal subchain $\{\bar{x}^i\}_{i>1}$ to target the desired distribution, is that the estimator $\widehat{L}(\bar{x})$ is unbiased. Otherwise, using for example a biased estimate, the marginal of the extended target (5) is not necessarily equal to the desired distribution.

3.1 Rendering of heterogeneous media

To evaluate the contribution function for a path, $\bar{x}^j$, in heterogenous participating media, we estimate the transmission between all vertices along the path using independent ratio trackings [Novák et al. 2014], providing us with unbiased estimators $\{\widehat{T}_i\}$. By combining these estimators, we can form an unbiased estimator of the total path contribution as

$$\widehat{L}_u(\bar{x}^j) = L_e^Y \left[\prod_{i=0}^{k-1} G(x_i^j, x_{i+1}^j) \widehat{T}_i(x_i^j, x_{i+1}^j)^Y \right] \left[\prod_{i=1}^{k-1} \rho(x_i^j)^Y \right]$$

where X^Y is the luminance of the spectrally valued function X. The resulting estimator $\widehat{L}_u(\bar{x}^j)$ is then unbiased as long as the ratio tracking estimators are unbiased and independent.

To propose new paths we use the perturbation strategies originally proposed by Veach and Guibas [1997] extended to handle participating media. To update the position of vertices in media, we use a proposal function similar to the medium perturbation proposed by Pauly *et al.* [2000] and later extended in the Mitsuba renderer [Jakob 2010]. Here, the medium vertex, x_i, is perturbed with uniform probability either forward or backward along the current edge, with a distance proportional to $\gamma\sigma_t(x_i)\exp(-\gamma\sigma_t(x_i))$, where γ is a scale parameter and $\sigma_t(x_i)$ is the extinction coefficient at the vertex. We consider both larger and smaller scales of the pertubations, i.e. we use a mixture of medium perturbations with different γs. For all results in this paper we used a mixture of $\gamma = 100$ and $\gamma = 50$. For the bidirectional path mutation, we use delta tracking to propose new path vertices in participating media. Note that, the intractable terms in the proposal density $q(\bar{x}^{i+1}|\bar{x}^i)$ cancels in the acceptance ratio (10), as long as the delta tracking is performed with respect to the luminance of the transmission. Our method is also compatible with recent perturbation proposals such as the manifold perturbation proposed by Wenzel and Marschner [Jakob and Marschner 2012].

4 Results

To evaluate the performance our new pseudo-marginal approach, we compare it to representative methods for efficient rendering of anisotropic heterogenous participating media and glossy transfer. All renderings were generated using the Mitsuba renderer [Jakob 2010].

(a) $M = 1$ (b) $M = 2$ (c) $M = 3$

Figure 4: *Equal time comparisons with varying number of averaged ratio trackings used to estimate the transmission between vertices along the path. Using a low M generally leads to more efficient estimators.*

In Figure 1, we show equal time comparisons for our pseudo-marginal ERPT method using ratio tracking, ERPT using Simpson quadrature ray-marching, and PSSMLT with ratio tracking. Note that, Raab *et al.* [2008] originally proposed to use PSSMLT with delta tracking to evaluate the transmission, however here we instead use the more recent ratio tracking estimator. For the ERPT algorithms, we use the manifold perturbation by Wenzel and Marschner [2012]. For ray-marching based MLT and ERPT, the default step-size selection implemented in Mitsuba was used. The glass cube contains a diffuse torus shape and an isotropic heterogenous media. Classical volumetric path tracing algorithms, such as bidirectional path tracing, produce very noisy results for this scene. Compared to ray-marching, the significanlty faster ratio tracking enables longer Markov chains to be used in the ERPT algorithm and a better exploration of the path space resulting in less image noise. The PSSMLT algorithm has problems finding good light paths in the scene, as it lacks the ability to use the efficient local path perturbations exploited by the ERPT algorithms.

Figure 3 shows equal time renderings of a scene from Veach [1997], filled with anisotropic scattering heterogenous media, ($g = 0.85$), stored as 64^3 voxels. As can be seen, this scene presents a difficult case for bidirectional path tracing. PSSMLT using ratio tracking performs slightly better, but also produces noisy results. Again, the pseudo-marginal MLT algorithm benefits from long Markov chains, efficient exploration of the path space, and outperforms previous methods in terms of image noise.

Parameter calibration An important practical consideration when using the pseudo-marginal MH update (10), is how many samples M should be used to compute the unbiased estimator $\widehat{L}_u(\bar{x})$. If a lower number of samples is used, the mixing of the marginal sub-chain, $\{\bar{x}^i\}_{i>1}$ can be poor. For example, if the the exact quantity is overestimated, the chain might get stuck at the current light path leading to a long series of rejected samples. A theoretically justified rule of thumb is to chose M so that the variance of $\log(\widehat{L}_u(\bar{x}))$ is approximately one, see e.g. Doucet *et al.* [2012]. However, this only holds when the noise in the estimate is independent with respect to the current light path $\bar{x}$, i.e. an assumption which often does not hold in practice. For most scenes where the MCMC based sampling methods provide a good alternative, i.e. scenes with complex light paths, we have found that using a small M is beneficial. The reason is that the noise introduced by an unbiased approximation is well compensated for by the faster exploration of the path space using longer Markov chains. A comparison of different settings for the number of averaged ratio trackings, M, is shown in Figure 4. For all the other result shown in this paper we used a single ratio tracking, $M = 1$, to evaluate the transmission.

5 Future work

Although the examples in this paper are based on replacing the intractable transmission term with unbiased estimators, the algorithm is general. It allows for any unbiased estimator of the scalar contribution function to be used in place of the corresponding exact quantity. This also opens up the possibility of replacing a known quantity with a more efficient approximated quantity, thus enabling more iterations of the underlying Markov chain in the same computational budget. An interesting venue for future work is to investigate applications of such procedures in rendering problems using e.g. expensive shading models.

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