An Exact Expression for the Average AoI in a Multi-Source M/M/1 Queueing Model

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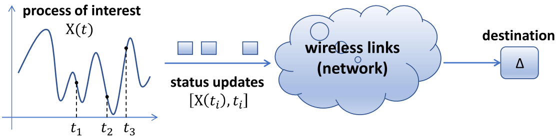
Background: Definition and Appliances

- Time sensitive information updates of a random process
  - Temperature of a specific environment (room, greenhouse, etc.)
  - A vehicular status (position, acceleration, etc.)
- A status update packet contains
  - The measured value of the monitored process
  - A time stamp representing the time when the sample was generated



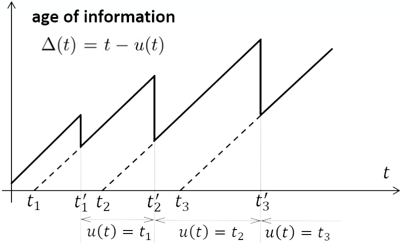
- One key enabler for these services is the freshness of the sensor's information at the destinations

Background: Age of Information

- The traditional metrics (throughput and delay) can not fully characterize the information freshness
- AoI is the time elapsed since the last received status update was generated, i.e., the random process

$$\Delta(t) = t - u(t) \tag{1}$$

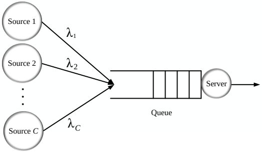
  - $u(t)$ is the time stamp of the most recently received update



- Average AoI: The most commonly used metrics for evaluating the AoI

System Model

- A set of independent sources
- The packets are generated according to the Poisson process
- An exponentially distributed service time $(1/\mu)$
- FCFS multi-source M/M/1 queueing model
- Two sources without loss of generality
- The average AoI of source 1[1]



$$\Delta_1 = \lambda_1 \left(\frac{\mathbb{E}[X_{1,i}^2]}{2} + \mathbb{E}[X_{1,i} S_{1,i}] + \mathbb{E}[X_{1,i} W_{1,i}] \right) \tag{2}$$

  - $X_{1,i}$ represents the ith interarrival time of source 1
  - $S_{1,i}$ represents the service time of packet $1, i$
  - $W_{1,i}$ represents the waiting time of packet $1, i$

[1] [R1] R. D. Yates and S. Kaul, "Real-time status updating: Multiple sources," in Proc. IEEE Int. Symp. Inform. Theory, Cambridge, MA, USA, Jul. 16, 2012, pp. 26662670.

AoI in a Multi-Source M/M/1 Queueing Model (1/9)

- The first term in (2): The interarrival time of source 1 follows the exponential distribution with parameter λ_1

$$\mathbb{E}[X_{1,i}^2] = 2/\lambda_1^2 \tag{3}$$

- The second term in (2): The interarrival time and service time of the packet $1, i$ are independent

$$\mathbb{E}[X_{1,i}S_{1,i}] = \mathbb{E}[X_{1,i}]\mathbb{E}[S_{1,i}] = \frac{1}{\mu\lambda_1} \tag{4}$$

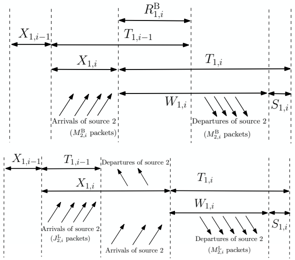
- The third term in (2) ($\mathbb{E}[X_{1,i}W_{1,i}]$): First we characterize the waiting time $W_{1,i}$ by means of two events $E_{1,i}^{\mathrm{B}}$ and $E_{1,i}^{\mathrm{L}}$ as

$$\begin{aligned} E_{1,i}^{\mathrm{B}} &= \{T_{1,i-1} \geq X_{1,i}\}, \\ E_{1,i}^{\mathrm{L}} &= \{T_{1,i-1} < X_{1,i}\} \end{aligned} \tag{5}$$

  - $T_{1,i} = S_{1,i} + W_{1,i}$, represents the system time of packet $1, i-1$

AoI in a Multi-Source M/M/1 Queueing Model (2/9)

- Event $E_{1,i}^{\mathrm{B}}$
- Event $E_{1,i}^{\mathrm{L}}$



$$W_{1,i} = \begin{cases} T_{1,i-1} - X_{1,i} + \sum_{i' \in \mathcal{M}_{2,i}^{\mathrm{B}}} S_{2,i'}, & E_{1,i}^{\mathrm{B}} \\ \sum_{i' \in \mathcal{M}_{2,i}^{\mathrm{L}}} S_{2,i'}, & E_{1,i}^{\mathrm{L}}, \end{cases} \tag{6}$$

- $\mathcal{M}_{2,i}^{\mathrm{B}}$: the set of packets of source 2 that must be served under $E_{1,i}^{\mathrm{B}}$
- $\mathcal{M}_{2,i}^{\mathrm{L}}$: the set of packets of source 2 that must be served under $E_{1,i}^{\mathrm{L}}$

AoI in a Multi-Source M/M/1 Queueing Model (3/9)

- The residual system time in the case $E^{\mathrm{B}}_{1,i}$

$$R^{\mathrm{B}}_{1,i} = T_{1,i-1} - X_{1,i}$$

- The total service time of source 2 packets in the case $E^{\mathrm{B}}_{1,i}$

$$S^{\mathrm{B}}_{1,i} = \sum_{i' \in \mathcal{M}^{\mathrm{B}}_{2,i}} S_{2,i'}$$

- The total service time of source 2 packets in the case $E^{\mathrm{L}}_{1,i}$

$$S^{\mathrm{L}}_{1,i} = \sum_{i' \in \mathcal{M}^{\mathrm{L}}_{2,i}} S_{2,i'}$$

- Consequently, the third term in (2) is given as

$$\mathbb{E}[X_{1,i}W_{1,i}] = \left(\mathbb{E}[R^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}] + \mathbb{E}[S^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}]\right)P(E^{\mathrm{B}}_{1,i}) + \mathbb{E}[S^{\mathrm{L}}_{1,i}X_{1,i}|E^{\mathrm{L}}_{1,i}]P(E^{\mathrm{L}}_{1,i}) \quad (7)$$

AoI in a Multi-Source M/M/1 Queueing Model (4/9)

- Calculation of $P(E^{\mathrm{B}}_{1,i})$

$$P(E^{\mathrm{B}}_{1,i}) = \int_0^\infty P(T_{1,i-1} \geq X_{1,i} | T_{1,i-1} = t) f_{T_{1,i-1}}(t) \mathrm{d}t \tag{8}$$

$$= \int_0^\infty F_{X_{1,i}}(t) f_{T_{1,i-1}}(t) \mathrm{d}t = 1 - \int_0^\infty e^{-\lambda_1 t} f_{T_{1,i-1}}(t) \mathrm{d}t = 1 - L_T(\lambda_1),$$

- $L_T(a)$ is given as

$$L_T(\lambda_1) = \frac{(1-\rho)\lambda_1 L_S(\lambda_1)}{\lambda_1 - \lambda(1 - L_S(\lambda_1))} \tag{9}$$

- $L_S(\lambda_1)$ is given as

$$L_S(\lambda_1) = \int_0^\infty \mu e^{-(\mu+\lambda_1)s} \mathrm{d}s = \frac{\mu}{\mu + \lambda_1} \tag{10}$$

- $E^{\mathrm{L}}_{1,i}$ is the complimentary event of $E^{\mathrm{B}}_{1,i}$

$$P(E^{\mathrm{L}}_{1,i}) = \frac{1-\rho}{1-\rho_2} \qquad P(E^{\mathrm{B}}_{1,i}) = 1 - P(E^{\mathrm{L}}_{1,i}) = \frac{\rho_1}{1-\rho_2} \tag{11}$$

AoI in a Multi-Source M/M/1 Queueing Model (5/9)

- Calculation of $\mathbb{E}[R^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}]$ (The first term in (7))

$$\mathbb{E}[R^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}] = \mathbb{E}[T_{1,i-1}X_{1,i}|E^{\mathrm{B}}_{1,i}] - \mathbb{E}[X^2_{1,i}|E^{\mathrm{B}}_{1,i}] \quad (12)$$
$$= \int_0^\infty \int_0^\infty xt f_{X_{1,i},T_{1,i-1}|E^{\mathrm{B}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty x^2 f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)\mathrm{d}x,$$

  - PDF $f_{X_{1,i},T_{1,i-1}|E^{\mathrm{B}}_{1,i}}(x,t)$

$$f_{X_{1,i},T_{1,i-1}|E^{\mathrm{B}}_{1,i}} = \begin{cases} 0 & x > t \\ \mu^2(1-\rho)(1-\rho_2)e^{-\lambda_1 x}e^{-\mu(1-\rho)t} & x \le t \end{cases} \quad (13)$$

  - PDF $f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x)$

$$f_{X_{1,i}|E^{\mathrm{B}}_{1,i}}(x) = \mu(1-\rho_2)e^{-\mu(1-\rho_2)x} \quad (14)$$

  - The first term in (7)

$$\mathbb{E}[R^{\mathrm{B}}_{1,i}X_{1,i}|E^{\mathrm{B}}_{1,i}] = \frac{1}{\mu^2(1-\rho_2)(1-\rho)} \quad (15)$$

AoI in a Multi-Source M/M/1 Queueing Model (6/9)

- Calculation of $\mathbb{E}[S_{c,i}^{\mathrm{B}_2} X_{c,i} | E_{c,i}^{\mathrm{B}}]$ (The second term in (7))

$$\mathbb{E}[S_{1,i}^{\mathrm{B}} X_{1,i} | E_{1,i}^{\mathrm{B}}] = \int_0^\infty x \mathbb{E}\big[\textstyle\sum_{i' \in \mathcal{M}_{2,i}^{\mathrm{B}}} S_{2,i'} | E_{1,i}^{\mathrm{B}}, X_{1,i} = x \big] f_{X_{1,i}|E_{1,i}^{\mathrm{B}}}(x) \mathrm{d}x$$
$$= \rho_2 \int_0^\infty x^2 f_{X_{1,i}|E_{1,i}^{\mathrm{B}}}(x) \mathrm{d}x \qquad (16)$$

  - PDF $f_{X_{1,i}|E_{1,i}^{\mathrm{B}}}(x)$

$$f_{X_{1,i}|E_{1,i}^{\mathrm{B}}}(x) = \mu(1-\rho_2) e^{-\mu(1-\rho_2)x} \qquad (17)$$

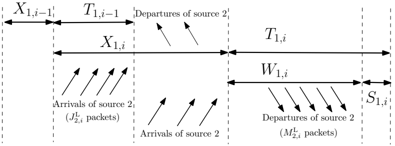
- The second term in (7)

$$\mathbb{E}[S_{1,i}^{\mathrm{B}} X_{1,i} | E_{1,i}^{\mathrm{B}}] = \frac{2\rho_2}{\mu^2(1-\rho_2)^2}$$

AoI in a Multi-Source M/M/1 Queueing Model (7/9)

- Calculation of $\mathbb{E}[S^{\mathrm{L}}_{1,i}X_{1,i}|E^{\mathrm{L}}_{1,i}]$ (The third term in (7))

$$\mathbb{E}[S^{\mathrm{L}}_{1,i}X_{1,i}|E^{\mathrm{L}}_{1,i}]=\int_0^\infty\int_0^\infty x\mathbb{E}[\sum_{i'\in\mathcal{M}^{\mathrm{L}}_{2,i}}S_{2,i'}|X_{1,i}=x,T_{1,i-1}=t,E^{\mathrm{L}}_{1,i}]\cdots$$
$$f_{X_{1,i}T_{1,i-1}|E^{\mathrm{L}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t=\frac{1}{\mu}\int_0^\infty\int_0^\infty x\mathbb{E}[M^{\mathrm{L}}_{2,i}|\cdots$$
$$X_{1,i}=x,T_{1,i-1}=t,E^{\mathrm{L}}_{1,i}]f_{X_{1,i}T_{1,i-1}|E^{\mathrm{L}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t$$
$$=\frac{1}{\mu}\int_0^\infty\int_0^\infty x\sum_{m=0}^{\infty}m\mathrm{Pr}[M^{\mathrm{L}}_{2,i}=m|X_{1,i}=x,\cdots$$
$$T_{1,i-1}=t,E^{\mathrm{L}}_{1,i}]f_{X_{1,i}T_{1,i-1}|E^{\mathrm{L}}_{1,i}}(x,t)\mathrm{d}x\mathrm{d}t, \qquad (18)$$



AoI in a Multi-Source M/M/1 Queueing Model (9/9)

- The third term is given by

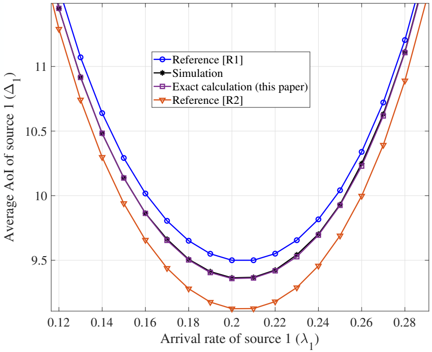
$$\mathbb{E}[W_{1,i}X_{1,i}|E_{1,i}^{\mathrm{L}}]=\frac{\lambda_1(1-\rho)}{P(E_{1,i}^{\mathrm{L}})}\int_0^{\infty}\int_0^{\infty}(t+\tau)e^{-\mu(t+\rho_1\tau)}\dots$$
$$\left(\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} m\bar{P}_{m|j}(\tau)\frac{(\lambda_2 t)^j}{j!}\right)\mathrm{d}\tau\mathrm{d}t \triangleq \frac{\lambda_1(1-\rho)}{P(E_{1,i}^{\mathrm{L}})}\Psi(\mu,\rho_1,\lambda_2) \quad (19)$$

- The average AoI of source 1

$$\Delta_1 = \lambda_1^2(1-\rho)\Psi(\mu,\rho_1,\lambda_2)+ \quad (20)$$
$$\frac{1}{\mu}\left(\frac{1}{\rho_1}+\frac{\rho}{1-\rho}+\frac{(2\rho_2-1)(\rho-1)}{(1-\rho_2)^2}+\frac{2\rho_1\rho_2(\rho-1)}{(1-\rho_2)^3}\right)$$

Results

- Exponential distribution with $\mu = 1$ and $\lambda_2 = 0.6$



- $[R1]$ M. Moltafet, M. Leinonen, and M. Codreanu, “Closed-form expression for the average age of information in a multi-source M/G/1 queueing model,” in Proc. IEEE Inform. Theory Workshop, Visby, Gotland, Sweden, Aug. 2528, 2019.
- $[R2]$ R. D. Yates and S. Kaul, “Real-time status updating: Multiple sources,” in Proc. IEEE Int. Symp. Inform. Theory, Cambridge, MA, USA, Jul. 16, 2012, pp. 26662670.

Thank You For Your Attention!

Questions?

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