Minimum-Perimeter Enclosures ⋆

Joseph S. B. Mitchell [1]

Department of Applied Mathematics and Statistics
Stony Brook University
Stony Brook, NY 11794-3600, USA

Valentin Polishchuk *

Helsinki Institute for Information Technology
University of Helsinki and Helsinki University of Technology
Helsinki, P.O.Box 68, FI-00014, Finland

Abstract

We give the first polynomial-time algorithm for the problem of finding a minimum-perimeter k-gon that encloses a given n-gon. Our algorithm is based on a simple structural result, that an optimal k-gon has at least one "flush" edge with the n-gon. This allows one to reduce the problem to computing the shortest k-link path in a simple polygon. As a by-product we observe that the minimum-perimeter "envelope" — a convex polygon with a specified sequence of interior angles — can also be found in polynomial time. Finally, we introduce the problem of finding optimal convex polygons restricted to lie in the region between two nested convex polygons. We give polynomial-time algorithms for the problems of finding the minimum restricted envelopes.

Key words: Computational Geometry, Polygon Optimization

⋆ Preliminary version appeared in Abstracts of the Fall Workshop on Computational and Combinatorial Geometry, Smith College, MA, USA, 10-11 November 2006.

* Corresponding author.

Email addresses: `jsbm@ams.sunysb.edu` (Joseph S. B. Mitchell), `valentin.polsihchuk@helsinki.fi` (Valentin Polishchuk).

URLs: `http://www.ams.sunysb.edu/~jsbm/` (Joseph S. B. Mitchell), `http://www.cs.helsinki.fi/u/polishch/` (Valentin Polishchuk).

[1] Partially supported by grants from NSF (CCF-0729019, CCF-0528209, CCF-0431030) and Metron Aviation (NASA Ames subcontract).

1 Introduction

A fundamental problem in geometric optimization is to compute a minimum-area or a minimum-perimeter convex k-gon (denoted Q^{A} or Q^{p}, resp.) that encloses a point set or, equivalently, a convex polygon $P = p_1, \ldots, p_n$. While efficient algorithms for computing Q^{A} have been known for 20 years [2, 3], the problem of computing Q^{p} has been repeatedly posed as an open [3, 4, 5, 6, 7, 9]. A linear-time algorithm is known for the case $k = 3$ [4].

Our algorithm for finding Q^{p} is based on a structural result about an optimal polygon: Local optimality implies that it is "flush" with P (Lemma 1). A polygon Q is said to be *flush* with an enclosed polygon P if an edge f of Q contains an edge e of P: $e \subseteq f$. The edge f (resp., e) is itself said to be flush with P (resp., with Q); we also say that e and f are flush. A vertex p of P is said to be a *rocking* point of an edge h of Q if p is the only contact point of h with P: $p = h \cap P$.

2 Computing Q^{p}

Our solution to the minimum-perimeter enclosing k-gon problem is based on the following lemma, whose (simple) proof we defer until the next subsection:

Lemma 1 *Q^{p} is flush with P.*

By Lemma 1, we can consider each edge of P as a candidate flush edge with Q^{p} and turn the scene into simple polygon $\overline{P}$ as in Figure 1. This reduces finding Q^{p} to solving n instances of the problem of finding a shortest $(k+1)$-link path in simple polygon $\overline{P}$, a problem which can be solved in polynomial time [9]. Thus we have our main result:

Theorem 2 *Q^{p} can be found in polynomial time.*

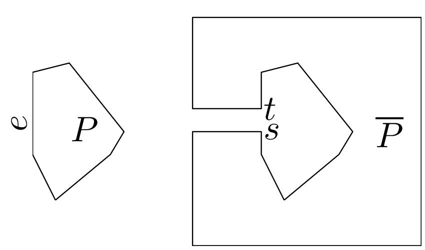

Fig. 1. Guess: e is flush with Q^{p}. Reduce to finding shortest $(k+1)$-link s-t path in the polygon $\overline{P}$. $\overline{P}$ must be large enough to accommodate Q^{p}; it is easy to see that $\overline{P}$ can, e.g., be taken to be the bounding box of P enlarged by a factor of 5 in all directions.

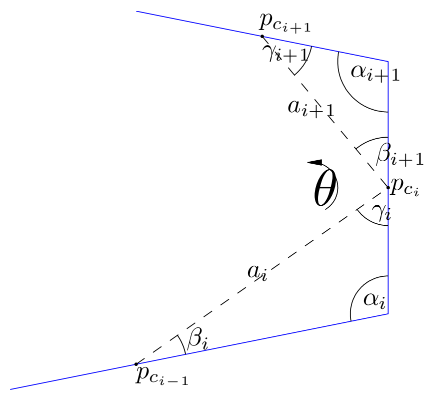

Fig. 2. $Q^{\mathsf{p}}(\theta)$ rocks on $\mathsf{C} = (p_{c_1}, \ldots, p_{c_k})$.

The Flushness Condition

We prove Lemma 1 with a "rotating calipers" [14] argument. [2]

PROOF of Lemma 1. Suppose not. By local optimality, each edge of Q^{p} rocks on a vertex of P. Let $\mathsf{C} = \{p_{c_1}, \ldots, p_{c_k}\} \subset \{p_1, \ldots, p_n\}$ be the rocking points of the edges of Q^{p}: $\mathsf{C} = bdQ^{\mathsf{p}} \cap P$. Let $\alpha_1, \ldots, \alpha_k$ be the sequence of angles of Q^{p}. Let β_i (resp.,γ_i) be the angles formed by the ith (resp., $i-1$st) side of Q^{p} and $p_{c_{i-1}}p_{c_i}$, let $a_i = |p_{c_{i-1}}p_{c_i}|$ (Fig. 2), where $i = 1 \ldots k$, and the subscripts are modulo k. By the Law of Sines the perimeter of Q^{p} is

$$\mathsf{p}(Q^{\mathsf{p}}) = \sum_{i=1}^{k} \frac{a_i}{\sin \alpha_i}(\sin \beta_i + \sin \gamma_i) = \sum_{i=1}^{k} \frac{a_i}{\sin \frac{\alpha_i}{2}} \cos \frac{\beta_i - \gamma_i}{2}$$

Start rotating each edge of Q^{p} around its rocking point counterclockwise by an angle θ; call the polygon formed by such a rotation $Q^{\mathsf{p}}(\theta)$. Let $\theta_{\min}, \theta_{\max}$ ($\theta_{\min} < 0 < \theta_{\max}$) be the angles at which $Q^{\mathsf{p}}(\theta)$ becomes flush with P. For $\theta \in [\theta_{\min}, \theta_{\max}]$, $Q^{\mathsf{p}}(\theta)$ is still a feasible k-gon, enclosing P; the perimeter of $Q^{\mathsf{p}}(\theta)$ as a function f of θ is

$$f(\theta) \equiv \mathsf{p}(Q^{\mathsf{p}}(\theta)) = \sum_{i=1}^{k} \frac{a_i}{\sin \frac{\alpha_i}{2}} \cos \frac{(\beta_i - \theta) - (\gamma_i + \theta)}{2} \tag{1}$$

Since $\beta_i - \theta$ and $\gamma_i + \theta$ are angles of a triangle, $|(\beta_i - \theta) - (\gamma_i + \theta)| < \pi$. Thus, as $\cos(\cdot)$ is a concave function on $(-\pi/2, \pi/2)$, each summand in (1) is a concave function of θ for $\theta \in (\theta_{\min}, \theta_{\max})$. Hence, $f(\theta)$ is also concave on $(\theta_{\min}, \theta_{\max})$ and attains its minimum at one of the ends of the interval, i.e., when $Q^{\mathsf{p}}(\theta)$ is flush with P.

[2] DePano [7] and Chang [6] proved the lemma for $k = 3$; the proof in [7] uses a complicated trigonometric argument, the proof in [6] is based on the similar result about minimum-*area* enclosing k-gon [8]. Our proof is different from those in [6, 7].

3 Extensions

The Running Time

In [9], two algorithms (call them Algorithms I and II) are presented for finding shortest (up to a multiplicative error of $1+\varepsilon$) k-link s-t path in a simple polygon. Naïvely, the running time of our solution in the previous section would be that of Algorithm I or II multiplied by n. We can improve it by looking into the details of the Algorithms.

Algorithm I, for each of $O(1/\sqrt{\varepsilon})$ equally-spaced orientations out of a vertex of the geodesic s-t path, extends a "greedy" min-link path out for k links in the direction of t; the obtained arrangement of $O(nk\frac{1}{\sqrt{\varepsilon}})$ segments is searched then for the approximate path. Since in our case any min-link path from a vertex of P to t (within $\overline{P}$) has at most 6 links, there are only $O(n\,O(1)\frac{1}{\sqrt{\varepsilon}})$ segments in the arrangement. Thus, the shortest k-link path in it can be found in $O(kn^2/\varepsilon)$ time. This must be done for each of the n choices of e.

Theorem 3 *The minimum-perimeter enclosing k-gon problem can be solved in $O(kn^3/\varepsilon)$ time.*

As an intermediate (and, actually, the time-dominating) step, Algorithm II computes all $O(n^2k)$ i-link paths between every pair of its possible flush edges ($i = 1 \ldots k$). After this is done in total $O(n^3k^3\log(Nk/\varepsilon^{\frac{1}{k}}))$ time [3] (Lemma 5.6 of [9]), the dynamic program (DP) is run to determine the best way to "split" k links in between the consecutive flush edges. In our setting, after edge e of P, flush with Q^*, is chosen, the DP will determine k values at each of the n edges of P with $O(nk)$ comparisons per value. Since there are n possible choices, all n DPs will run in $O(n \cdot nk \cdot nk)$ time. This is still dominated by the $O(n^3k^3\log(Nk/\varepsilon^{\frac{1}{k}}))$ time to tabulate the i-link paths. Thus,

Theorem 4 *The minimum-perimeter enclosing k-gon problem can be solved in $O\left(n^3k^3\log(\frac{Nk}{\varepsilon^{1/k}})\right)$ time.*

Minimum-perimeter envelope

The works [8, 10] considered the problem of finding the minimum *envelope* — an enclosing k-gon with a specified sequence $A = (\alpha_1, \ldots, \alpha_k)$ of angles. The algorithms in [8, 10] for finding the minimum-*area* envelope Q_A^{A} are based (unsurprisingly) on the flushness condition. Note that our Flushness Lemma (Lemma 1) actually shows that the minimum-*perimeter* envelope Q_A^{p} is also

[3] N is the largest integer in the problem input.

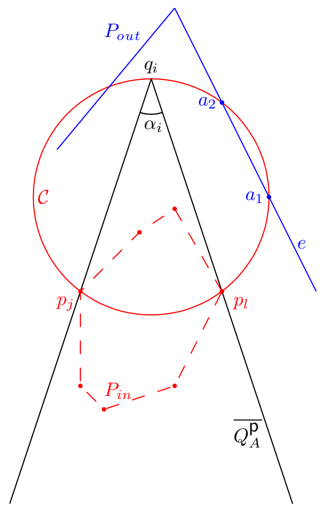

Fig. 3. If $q_i \notin \{a_1, a_2\}$, q_i can be moved both clockwise and counterclockwise around $\mathcal{C}$; moving in one of the directions decreases the perimeter of $\overline{Q_A^{\mathsf{P}}}$.

flush with P and thus, can be found in polynomial time:[4]

Lemma 5 Q_A^{P} *can be computed in* $O(n^2k)$ *time.*

PROOF. By the flushness condition, we can consider each of the nk pairs of an edge of P and an edge of Q_A^{P} as a candidate flush pair; for a given pair of edges, in $O(n)$ time we can determine the other points of contact of the envelope with P and compute the perimeter of the envelope.

The running time in Lemma 5 can be improved by following the algorithm of [10] for finding Q_A^{A}. The $O(nk \log k)$ running time of the algorithm is due to exploiting the rotating calipers idea [14]. Instead of recomputing the area for each of the nk flush edge pairs, the (expression that gives the) area is updated as the envelope "rolls" around P; the flushness condition ensures that only the nk flush envelopes need to be tested for optimality. The approach can be extended verbatim to the perimeter minimization case. Thus,

Theorem 6 *The minimum-perimeter envelope can be found in* $O(nk \log k)$ *time.*

[4] Chang [6, p. 5] and Schwarz et al. [13] mention that [8] solves the *perimeter*-minimization version as well. In fact, perimeter minimization for the general case of arbitrary A (not just $\alpha_i = \frac{k-2}{k}\pi$) is not mentioned in [8] anywhere but in the abstract. We believe (after looking thoroughly into the details of [8]) that the paper did not claim an algorithm for finding Q_A^{P} for arbitrary A; see Appendix for details.

In the original statement of the minimum enclosure problem, the vertices of the enclosure are allowed to be placed anywhere in the plane. We propose a generalization, in which two nested convex polygons P_{out} and $P_{in} \subset P_{out}$ are given, and a minimum k-gon restricted to lie in between P_{in} and P_{out} is sought; denote the minimum-area and minimum-perimeter restricted k-gons by $\overline{Q^{\mathsf{A}}}$ and $\overline{Q^{\mathsf{p}}}$, respectively. The problem is motivated by a classification problem that seeks a low-complexity separator between data points of two types.

We give polynomial-time algorithms for finding minimum-area and minimum-perimeter restricted *envelopes* $\overline{Q_A^{\mathsf{A}}}$ and $\overline{Q_A^{\mathsf{p}}}$. Our solution is based on the fact that the optimal restricted polygons are "either flush or bash".

Following [9] we say that $\overline{Q_A^{\mathsf{p}}}$ is *bash* with P_{out} if a vertex of the former belongs to a boundary edge of the latter. In the next two lemmas we show that $\overline{Q_A^{\mathsf{p}}}$ is either flush with P_{in} or is bash with P_{out}, and that there is only a polynomial number of possible locations for the bash points.

Lemma 7 *$\overline{Q_A^{\mathsf{p}}}$ is either flush with P_{in} or is bash with P_{out}.*

PROOF. Suppose not. As in the proof of Lemma 1, start rotating each edge of the envelope around the vertex of P_{in}, on which it rocks, by the same angle — the rotated envelope still has the angle sequence A and its perimeter as a function of the turn angle is unimodal. Thus, $\overline{Q_A^{\mathsf{p}}}$ can be rotated in one of the directions, decreasing its perimeter, until either it becomes flush with P_{in} or bash with P_{out} (Fig. 3).

Lemma 8 *Suppose that a bash vertex q_i of $\overline{Q_A^{\mathsf{p}}}$ and the edge $e \ni q_i$ of P_{out} are given. Suppose that the edges $q_{i-1}q_i$ and q_iq_{i+1} of $\overline{Q_A^{\mathsf{p}}}$ rock on the vertices p_j and p_l of P_{in}. Let $\mathcal{C}$ be the circle through p_j, p_l such that the segment p_jp_l is seen at the angle α_i from the points on $\mathcal{C}$; let a_1, a_2 be the points of intersection (if any) of $\mathcal{C}$ with e. Then either $q_i = a_1$ or $q_i = a_2$ (Fig. 3).*

PROOF. Otherwise, q_i can be moved both clockwise and counterclockwise around $\mathcal{C}$ while $\overline{Q_A^{\mathsf{p}}}$ stays a feasible restricted envelope; moving in one of the directions strictly decreases the perimeter of $\overline{Q_A^{\mathsf{p}}}$.

Theorem 9 *$\overline{Q_A^{\mathsf{p}}}$ can be computed in $O(n_{in}^3 n_{out} k)$ time, where n_{in} and n_{out} are the complexities of P_{in} and P_{out}.*

PROOF. If $\overline{Q_A^{\mathsf{P}}}$ is flush with P_{in}, find $\overline{Q_A^{\mathsf{P}}}$ in $O(n_{in}^2 k) = O(n_{in}^3)$ time as in the proof of Lemma 5. Otherwise, by Lemma 8, there is at most $2\binom{n_{in}}{2} n_{out} k = O(n_{in}^2 n_{out} k)$ possible locations for the bash points of $\overline{Q_A^{\mathsf{P}}}$ (and each of them can be found in constant time by intersecting the circle and the edge of P_{out}); for each choice of the triple (p_j, p_l, q_i), $\overline{Q_A^{\mathsf{P}}}$ can be found in additional $O(n_{in})$ time by wrapping the envelope around P_{in}.

Similarly to Lemma 1, Mount and Silverman [10] showed that the *area* of the envelope as a function of the turn angle is unimodal. Thus, the above algorithm also works for finding $\overline{Q_A^{\mathsf{A}}}$.

4 Discussion

It may be possible to improve the running times of our algorithms by giving a finer characterization of optimal enclosures, in hopes that fewer edges have to be tested for flushness. It would be interesting to investigate how the minimum-area and minimum-perimeter enclosing k-gons differ; e.g., what bounds can be shown on the Fréchet distance between them?

A related problem is that of computing a shortest *external watchman route* for a convex polygon P: Determine a shortest k-link closed route such that a guard following the route sees the boundary of P from outside. As in the case of (unrestricted) external watchman routes, not restricted to k links, an optimal route is either a closed loop or a doubled path [1, 11]. The shortest closed loop is the minimum-perimeter enclosing k-gon. It remains open how to find a shortest $\lfloor k/2 \rfloor$-link path whose doubling is an external watchman route.

Acknowledgments

We thank David Mount for helpful discussions and the anonymous referees for their feedback.

References

[1] R. Absar and S. Whitesides. On computing shortest external watchman routes for convex polygons. In *Proceedings of the 18th Canadian Confer-*

ence on Computational Geometry (CCCG'06), pages 59–62, 2006.

[2] A. Aggarwal, J. S. Chang, and C. K. Yap. Minimum area circumscribing polygons. *Visual Computer*, 1:112–117, 1985.

[3] A. Aggarwal and J. K. Park. Notes on searching in multidimensional monotone arrays. In *Proceedings of the 29th Annual IEEE Symposium on Foundations of Computer Science*, pages 497–512, 1988.

[4] B. K. Bhattacharya and A. Mukhopadhyay. On the minimum perimeter triangle enclosing a convex polygon. In J. Akiyama and M. Kano, editors, *Discrete and Computational Geometry, Japanese Conference, JCDCG 2002, Tokyo, Japan, December 6-9, 2002, Revised Papers*, Lecture Notes in Computer Science 2866, pages 84–96. Springer, 2003.

[5] J. E. Boyce, D. P. Dobkin, R. L. Drysdale, III, and L. J. Guibas. Finding extremal polygons. *SIAM Journal on Computing*, 14:134–147, 1985.

[6] J. S. Chang. Polygon optimization problems. Report 240, CS, NYU, New York, NY, 1986.

[7] N. A. A. DePano. *Polygon approximation with optimized polygonal enclosures: applications and algorithms*. Ph.D. thesis, Department of Computer Science, Johns Hopkins University, 1987.

[8] N. A. A. DePano and A. Aggarwal. Finding restricted k-envelopes for convex polygons. In *Procedings of the 22nd Allerton Conference on Communication, Control, and Computing*, pages 81–90, 1984.

[9] J. S. B. Mitchell, C. Piatko, and E. M. Arkin. Computing a shortest k-link path in a polygon. In *Procedings of the 33rd Annual IEEE Symposium on Foundations of Computer Science*, pages 573–582, 1992.

[10] D. M. Mount and R. Silverman. Minimum enclosures with specified angles. Technical Report CS-3219, University of Maryland, 1994.

[11] S. Ntafos and L. Gewali. External watchman routes. *Visual Computer*, 10:474–483, 1994.

[12] H. Pirzadeh. Computational geometry with the rotating calipers. Master's thesis, McGill University, 1999.

[13] C. Schwarz, J. Teich, A. Vainshtein, E. Welzl, and B. L. Evans. Minimal enclosing parallelogram with application. In *Proceedings of the 11th Annual ACM Symposium on Computational Geometry*, pages C34–C35, 1995.

[14] G. T. Toussaint. Solving geometric problems with the rotating calipers. In *Proceedngs of IEEE MELECON '83*, pages 1–4, 1983.

A Some Notes on DePano and Aggarwal's Work

One of the contributions of this paper is an algorithm for finding minimum-perimeter enclosing envelope. Since several authors—e.g., Chang [6, p. 5] and Schwarz et al. [13]—mention that DePano and Aggarwal [8] solved the problem in their 1984 paper, we feel obligated to explain why we believe that might not be the case.

(1) DePano and Aggarwal [8] provide details only for the minimum-area *regular* enclosing k-gon problem for $k = 3, 4$. The algorithms in the paper are based on the flushness lemma. After it is proved, the authors state that all that needed is to test each edge of P as a candidate flush edge (which takes $O(k)$ time — "compute the minimum from among k angles and compute the k-gon's area" [8, p. 85]). This enables the authors to claim an $O(nk)$-time solution. In fact, it does not matter which side of a *regular* k-gon is flush with P (since all the sides of a regular polygon are "the same"), all that matters is which side of P is flush with the optimal k-gon. For an "irregular" k-gon though, it matters *both* which side of P is flush with the k-gon *and* which side of the k-gon is flush with P; thus, in the general case, nk pairs of candidate flush edges must be tested.

(2) DePano and Aggarwal [8] consider *area*-minimization for the most of the paper. The only place in the paper where the solution to the minimum-*perimeter* envelope is claimed is the abstract: "...for any class of k-gons with fixed angle sequence, we achieve an $O(nk)$ algorithm. All of these results extend to minimal perimeter enclosing k-gons."

First of all (see above) we question the $O(nk)$ running time even for the case of the *area* minimization. Second, in Section 3, Summary of Results, the authors state their contribution only to the case of minimum-perimeter *regular* k-gons: "The restriction to regular k-envelopes has an added bonus in that the regular k-envelope also gives the minimum perimeter regular k-gonal enclosure." Finding irregular minimum-perimeter enclosures is not mentioned! Section 6 (Minimum Perimeter Regular and Equiangular k-Gons), the (only!) section that describes possible extension of the algorithms to the *perimeter* minimization, does not mention irregular minimum-perimeter enclosures either. Thus, the only relation of [8] to perimeter minimization is the observation that the minimum area implies minimum perimeter. (The observation is, in fact, faulty as Fig. 3.2 in [12] shows.) This makes us think that, contrary to the statements in the abstract of [8], in [6, p. 5], and in [13], DePano and Aggarwal did not claim the solution to the minimum-*perimeter* enclosing envelope problem.