Segment Watchman Routes*

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Abstract

We consider a variant of the 2-watchmen problem that ensures that every point in a polygon $\mathbf{P}$ is seen from more than one direction: we search for routes W_1, W_2, such that for each $p \in \mathbf{P}$ there exist $w_1 \in W_1, w_2 \in W_2$ that see p and such that $p \in \overline{w_1 w_2} \subset \mathbf{P}$. We show that finding the two routes that are optimal with respect to the min-max criterion is NP-hard in simple polygons and present a 2-approximation algorithm for this case; moreover, we provide a polynomial-time algorithm for computing the two optimal routes with respect to the min-sum criterion in convex polygons. Finally, we discuss a generalized version of the problem with more than two watchmen.

1 Introduction

In the classical Watchman Route Problem, introduced by Chin and Ntafos [3, 4], we ask for the shortest route inside a given simple polygon $\mathbf{P}$, such that all points of $\mathbf{P}$ are visible from at least one point on the route (this can be solved in polynomial time [14, 15]). In this context, a point $p \in \mathbf{P}$ *sees* another point $q \in \mathbf{P}$ if the line segment $\overline{pq}$ is fully contained in $\mathbf{P}$.

Carlsson et al. [2] raised the m-watchmen problem as a natural generalization: we are given m watchmen (with or without given starting points) for which we aim to find routes, such that each point in $\mathbf{P}$ is visible from at least one of the m routes. Two common objectives for this problem are to minimize the total length of all m watchman routes (called *min-sum*) and to minimize the length of the longest route assigned to any watchman (called *min-max*).

When considering m watchmen, we only require each point to be seen at least once, without any guarantees on any kind of robustness. However, in practice, we may aim to make our routes robust against potential issues. For example, one or more watchmen may fail, especially in remote regions. Additionally, observing a point from multiple angles can improve observation quality. This is crucial to make the theoretically intriguing routes applicable for real-world scenarios. In this paper, we aim to enhance the coverage quality by guaranteeing a point to be seen from multiple directions.

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Problem Definition. Let $\mathbf{P}$ be a *simple polygonal domain.* A route W in $\mathbf{P}$ is called a *watchman route* if every point in $\mathbf{P}$ is visible from some point on W, and we denote its length by $|W|$. A point $p \in \mathbf{P}$ is *segment-guarded* by two points $w_1, w_2 \in \mathbf{P}$ if p lies on the line segment $\overline{w_1w_2}$, and p is visible to both w_1 and w_2, i.e., $\overline{w_1w_2}$ is fully contained in $\mathbf{P}$ (while the watchmen do not need to be at w_1 and w_2 at the same time).

Two routes W_1, W_2 in $\mathbf{P}$ are *segment watchman routes* for $\mathbf{P}$ if for every point $p \in \mathbf{P}$ there exist two points $w_1 \in W_1$ and $w_2 \in W_2$ such that p is segment-guarded by w_1 and w_2. We consider the following two problems:

▷ Problem 1 (Min-Max Segment Watchmen). Given a polygonal domain $\mathbf{P}$, find segment watchman routes W_1, W_2 such that $\max_i |W_i|$ is minimized.

▷ Problem 2 (Min-Sum Segment Watchmen). Given a polygonal domain $\mathbf{P}$, find segment watchman routes W_1, W_2 such that $\sum_i |W_i|$ is minimized.

In the same manner, we define a point $p \in \mathbf{P}$ to be *triangle-guarded* (or *k-gon-guarded*) if there exist points w_i on routes W_i, $i = 1, 2, 3$ (or $i = 1, \dots, k$), such that the segments $\overline{w_ip}, \forall i$, are fully contained in $\mathbf{P}$ and do not share a point other than p. With this, we define the related min-max and min-sum optimization problems analogously to Problems 1 and 2.

Note that, due to limited space, we omit the proofs of statements marked by ($\star$).

Related Work. Carlsson et al. [2] showed that the m-watchmen problem is NP-hard in simple polygons and provided a polynomial time algorithm for histograms. Polynomial time algorithms for different polygon classes, using either the min-sum or the min-max objective, have also been presented in [1, 9, 11, 12]. Recently, Nilsson and Packer [10] proposed a 5.969-approximation algorithm to compute min-max 2-watchman routes in simple polygons.

The robustness requirement we employ for watchman routes in this paper is closely related to the problems of two-sensor visibility and triangle guarding for stationary guards introduced by Efrat et al. [6] and Smith and Evans [13], respectively. Both considered two polygons $\mathbf{Q}, \mathbf{P}$ with $\mathbf{Q} \subseteq \mathbf{P}$, where the subpolygon $\mathbf{Q}$ should be guarded by guards placed in $\mathbf{P}$ (assuming that $\mathbf{Q}$'s boundary is transparent). For Efrat et al. a point $p \in \mathbf{Q}$ is *2-guarded at angle* α by two guards g_1, g_2 if $\angle g_1pg_2 \in [\alpha, \pi - \alpha]$ and both guards see p. Smith and Evans defined a point $p \in \mathbf{Q}$ to be *triangle-guarded* by g_1, g_2, g_3 if p is seen by each of the three guards and is contained in the triangle spanned by them. Another variant of robust guarding has recently been established by Das et al. [5]; and a variant of robustness for a single watchman by Langetepe et al. [8].

2 Preliminaries and Key Lemma

Let $\mathbf{P}$ be a simple polygon with n vertices. We assume that $\mathbf{P}$ does not contain vertices with an internal angle of exactly 180°, i.e., no three consecutive vertices are on the same line. If $\mathbf{P}$ does contain such a vertex, we can simply remove it.

Let W_1, W_2 be segment watchman routes for $\mathbf{P}$. From the definition, we obtain:

▶ **Observation 2.1.** *Each of W_1 and W_2 is a watchman route for $\mathbf{P}$.*

▷ Claim 2.2. Every convex vertex of $\mathbf{P}$ is visited by one of W_1 or W_2.

Proof. Let v be a convex vertex of $\mathbf{P}$. Then v lies on a line segment $\overline{w_1w_2}$ with $w_1 \in W_1$ and $w_2 \in W_2$, and the segments $\overline{vw_1}, \overline{vw_2}$ are contained in $\mathbf{P}$. As the interior angle at v is strictly smaller than 180°, any line segment in $\mathbf{P}$ that contains v has v as one of its endpoints. ◁

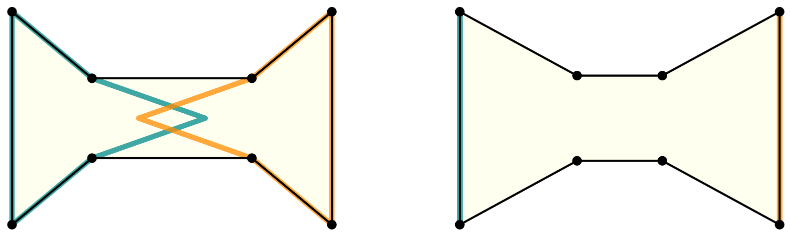

Figure 1 Min-max segment watchman routes may or may not need to overlap.

We now establish sufficient conditions for two routes to be segment watchman routes; an example is illustrated in Figure 1.

▶ **Lemma 2.3** (The Conditions Lemma). *Two routes W_1 and W_2 are segment watchman routes for* $\mathbf{P}$ *if the following conditions hold:*

1. *Every convex vertex is visited by one of W_1 or W_2.*
2. *Both W_1 and W_2 visit the visibility polygon of each convex vertex.*
3. *Both W_1 and W_2 are simple and relatively convex (i.e., a route does not cross itself, and for any two points inside the region enclosed by the route, their shortest path is also contained within the enclosed region).*

Proof. First, we show that Condition 2 implies that W_1 and W_2 are watchman routes. Assume that there is a point $p \in \mathbf{P}$ that is not seen by W_i, i.e., no point of W_i lies in p's visibility polygon. Hence, W_i is fully contained in one of the *pockets* $\mathbf{P}'$ of p's visibility polygon (a subpolygon of $\mathbf{P}$ in which no point is visible from p). Extend the pocket's *window* w (the line segment that separates $\mathbf{P}'$ and $\mathbf{P} \setminus \mathbf{P}'$) into a maximal line segment ℓ contained in $\mathbf{P}$. Without loss of generality, let ℓ be a vertical line segment with $\mathbf{P}'$ to its right. As $p \in \ell$, $\ell \setminus w$ is either a polygonal edge with a convex endpoint not seen by W_i, or it splits $\mathbf{P}$ into at least two subpolygons; see Figure 2. At least one of the subpolygons, say $\mathbf{P}''$, also lies to the right of ℓ. W_i cannot see any convex vertex in $\mathbf{P}''$, yielding a contradiction.

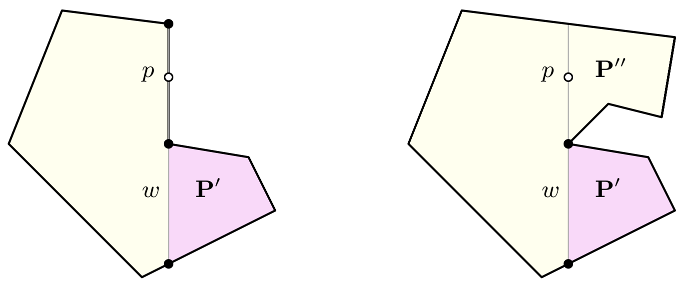

Figure 2 $\ell \setminus w$ is either an edge of $\mathbf{P}$ with a convex endpoint (left), or it splits $\mathbf{P}$ into at least two subpolygons, one of which also lies to the right of ℓ (right).

We now show that Conditions 1–3 imply that W_1 and W_2 are segment watchman routes. Consider a point $p \in \mathbf{P}$. Since both W_1 and W_2 are watchman routes, there exists at least one point on W_1 and at least one point on W_2 that p sees. Consider the two wedges defined by the angles from which p is viewing W_1 and W_2, as visualized in Figure 3: let F_1 be the maximal wedge bounded by two rays starting at p, such that for every ray ρ in F_1 there is a point $w \in W_1$ in this direction that p sees. Note that because $\mathbf{P}$ is simple, F_1 is a single wedge. The wedge F_2 is defined analogously for W_2.

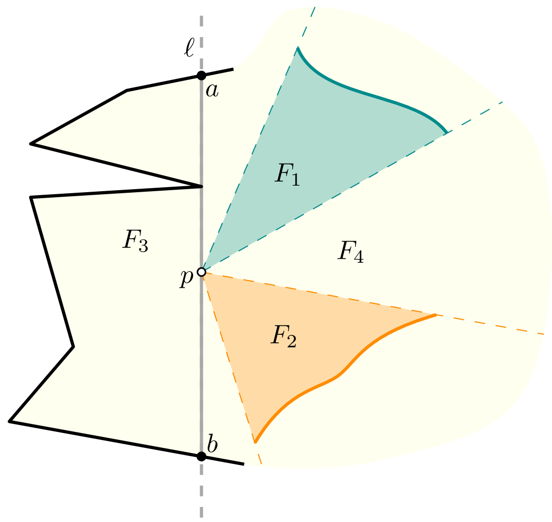

Figure 3 The wedges F_1 and F_2 define the angles from which p is viewing W_1 and W_2, respectively. If F_3 or F_4 is larger than 180°, then there is a convex vertex on the left side of ℓ which is not visited.

Each of the two wedges F_1, F_2 covers either 360° (if p lies on or within the relatively convex route) or less than 180° (because both routes are relatively convex). If at least one of F_1, F_2 covers 360° around p, then p is segment-guarded: assume that F_2 covers 360° around p, and let w_1 be a point on W_1 that sees p. Then the ray from w_1 in the direction of p intersects W_2 at point w_2 that sees p, and thus p is segment-guarded by $\overline{w_1 w_2}$.

Hence, assume that neither F_1 nor F_2 covers 360° around p. Let F_3 (and possibly F_4) be the maximal wedge(s) bounded by two rays starting at p, such that for every ray ρ in F_3 (and F_4) there is no point $w \in W_1$ or $w \in W_2$ in this direction that p sees. Then the plane around p can be split into up to four wedges, depending on whether F_1 and F_2 intersect: F_1, F_2, F_3 and F_4; or F_1, F_2, F_3, and one wedge with the overlap of F_1 and F_2.

We argue that neither F_3 nor F_4 can cover more than 180°. Without loss of generality, assume that F_3 covers more than 180°. Consider a line ℓ through p in F_3 that does not contain an edge of the boundary of $\mathbf{P}$, and assume that ℓ is a vertical line and that both F_1 and F_2 are on the right side of ℓ. Let $\overline{ab}$ be the maximal line segment on ℓ that is contained in $\mathbf{P}$. Then $\overline{ab}$ splits $\mathbf{P}$ into at least two subpolygons, and at least one of them, $\mathbf{P}'$, is on the left side of $\overline{ab}$. Because $\mathbf{P}$ is simple and both W_1 and W_2 do not cross $\overline{ab}$, there are no points of W_1 and W_2 in $\mathbf{P}'$. However, $\mathbf{P}'$ must contain a convex vertex v. This yields a contradiction, as by Condition 1, v needs to be visited by at least one of the watchman routes. ◀

We define the *relative convex hull* of a route in a simple polygon $\mathbf{P}$ as the simple polygon $\mathbf{Q}$ such that, for any two points inside the region enclosed by the route, the geodesic connecting them is also contained within $\mathbf{Q}$. Specifically, we refer to the boundary of $\mathbf{Q}$ as the relative convex hull. Hence, if a route is relatively convex, it coincides with its relative convex hull.

In Lemma 2.3, the conditions imply that W_1 and W_2 are segment watchman routes. However, there exist segment watchman routes that do not fulfill these conditions, see Figure 4.

On the other hand, we obtain an if-and-only-if statement for optimal watchman routes:

▶ **Observation 2.4.** *Let $\mathbf{P}$ be a simple polygon. Two routes W_1 and W_2 are optimal segment watchman routes for $\mathbf{P}$, if and only if the conditions from Lemma 2.3 hold.*

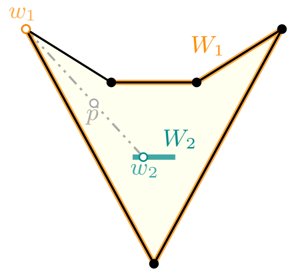

Figure 4 W_1 and W_2 are segment watchman routes (e.g., p lies on $\overline{w_1 w_2}$), but do not fulfill the conditions of Lemma 2.3. They are not optimal, e.g., W_1's relative convex hull (in this case the boundary of the polygon) is shorter than W_1, and this relative convex hull together with W_2 are segment watchman routes.

3 Min-Max Segment Watchman Routes in Simple Polygons

We sketch a reduction showing that the problem is NP-hard even in simple polygons. Complementarily, we provide a polynomial-time 2-approximation.

3.1 Computational Complexity

We reduce from Multiway Number Partitioning [7]. In particular, for our purposes, we ask to partition a set of numbers into two sets of equal sum; also referred to as Partition, which is known to be weakly NP-hard.

▶ **Theorem 3.1** (⋆). *Problem 1 is* NP*-hard even in simple polygons.*

Proof sketch. Construct a star-shaped polygon as in Figure 5. The length of a spike's boundary (i.e., the path $v_{2i-1}, v_{2i}, v_{2i+1}$) represents the value α_i from the Partition instance φ, and let T denote the sum of all values. Both watchmen start in the bottommost convex vertex v_0, and thus need to return to it. It is easy to see that a min-max segment watchman route of length $T/2 + \varepsilon$ exists iff there exist a partition of φ into two sets of equal sum. ◀

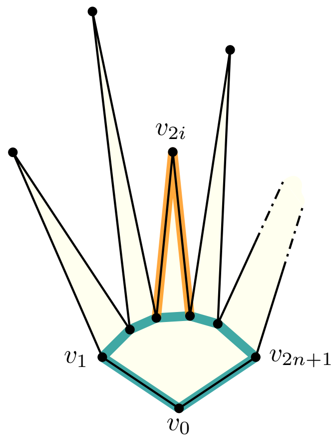

Figure 5 High-level idea of the type of polygon utilized in the NP-hardness reduction.

3.2 Approximation Algorithm

Let k be the number of convex vertices of a given polygon $\mathbf{P}$. We enumerate the convex vertices in counterclockwise order $v_0, \ldots, v_{k-1}$, with v_0 chosen arbitrarily. In the following, we assume, without loss of generality, that indices are counted modulo k.

Let v_i and v_j be two different convex vertices and let C_{ij} be the shortest route that visits the convex vertices $v_i, \ldots, v_{j-1}$. C_{ji} is then the shortest route that visits the convex vertices $v_j, \ldots, v_{i-1}$. Clearly, C_{ij} and C_{ji} can be computed in linear time. Let $C_{\mathbf{P}}$ be the shortest route that visits all the convex vertices of $\mathbf{P}$. $C_{\mathbf{P}}$ can also be computed in linear time.

Let D_{ij} be the shortest route that starts and ends at v_i, and that sees all the convex vertices $v_j, \ldots, v_{i-1}$. The route D_{ji} is then the shortest route that starts and ends at v_j, and that sees all the convex vertices $v_i, \ldots, v_{j-1}$. Each of D_{ij} and D_{ji} can be computed in $\mathcal{O}(n^3)$ time by modifying the algorithm of Jiang and Tan [15]. Let $D_{\mathbf{P}}$ be the shortest floating watchman route in $\mathbf{P}$ (that is, the shortest watchman route without a given starting point). We can compute $D_{\mathbf{P}}$ in $\mathcal{O}(n^4)$ time [14, 15].

Let $\mathrm{H}(T)$ denote the relative convex hull of a route T in $\mathbf{P}$. We define $W_{ij} \overset{\text{def}}{=} \mathrm{H}(C_{ij} \cup D_{ij})$, connecting the two routes at v_i and taking the relative convex hull of them.

We construct our approximate solution by choosing the pair

$$(W_1, W_2) = \underset{i \neq j}{\arg\min} \left\{ \max\{|W_{ij}|, |W_{ji}|\}, \max\{|C_{\mathbf{P}}|, |D_{\mathbf{P}}|\} \right\}.$$

By Lemma 2.3, (W_1, W_2) is a feasible solution for the segment watchman routes problem. Denote by $\mathrm{OPT}(\mathbf{P})$ the size of an optimal solution for $\mathbf{P}$. We claim the following result.

▶ **Theorem 3.2.** $\max\left\{|W_1|, |W_2|\right\} \leq 2 \cdot \mathrm{OPT}(\mathbf{P})$.

Proof. Let W_1^* and W_2^* be two segment watchman routes with $\max\left\{|W_1^*|, |W_2^*|\right\} = \mathrm{OPT}(\mathbf{P})$. Without loss of generality, we may assume that W_1^* and W_2^* are as short as possible.

If W_1^* or W_2^* visits all convex vertices of $\mathbf{P}$, then $(C_{\mathbf{P}}, D_{\mathbf{P}})$ is an optimal solution to the problem and the theorem therefore holds. Hence, for the remainder of this proof, we assume that W_1^* visits some fixed convex vertex v_i and W_2^* visits a different fixed convex vertex v_j.

Since W_1^* visits v_i and it either sees or visits the convex vertices $v_j, \ldots, v_{i-1}$ by construction, we have that $|D_{ij}| \leq |W_1^*|$. Similarly, W_2^* visits v_j and it either sees or visits the convex vertices $v_i, \ldots, v_{j-1}$, yielding $|D_{ji}| \leq |W_2^*|$. We distinguish the following cases.

W_1^* and W_2^* do not intersect. Because W_1^* and W_2^* do not intersect, the two convex vertices v_i and v_j can be chosen so that W_1^* visits $v_i, \ldots, v_{j-1}$ by increasing index (modulo k) and sees the remaining ones, whereas W_2^* visits $v_j, \ldots, v_{i-1}$ and sees the remaining ones. From this, it follows that $|C_{ij}| \leq |W_1^*|$ and $|C_{ji}| \leq |W_2^*|$. We obtain that

$$\begin{aligned} \max\left\{|W_1|, |W_2|\right\} &\leq \max\left\{|\mathrm{H}(C_{ij} \cup D_{ij})|, |\mathrm{H}(C_{ji} \cup D_{ji})|\right\} \\ &\leq \max\left\{2|W_1^*|, 2|W_2^*|\right\} = 2 \cdot \max\left\{|W_1^*|, |W_2^*|\right\} = 2 \cdot \mathrm{OPT}(\mathbf{P}). \end{aligned}$$

W_1^* and W_2^* intersect. Because W_1^* and W_2^* intersect and together visit all the vertices, we have $|C_{\mathbf{P}}| \leq |W_1 \cup W_2| = |W_1^*| + |W_2^*|$ and $|D_{\mathbf{P}}| \leq \min\{|W_1^*|, |W_2^*|\}$, as both W_1^* and W_2^* are watchman routes. We obtain that

$$\begin{aligned} \max\left\{|W_1|, |W_2|\right\} &\leq \max\left\{|C_{\mathbf{P}}|, |D_{\mathbf{P}}|\right\} \leq \max\left\{|W_1^*| + |W_2^*|, \min\{|W_1^*|, |W_2^*|\}\right\} \\ &\leq 2 \cdot \max\left\{|W_1^*|, |W_2^*|\right\} = 2 \cdot \mathrm{OPT}(\mathbf{P}). \end{aligned}$$ ◀

In fact, we may also let $W_2 = C_{\mathbf{P}}$ to avoid computing a floating shortest watchman route. The proof also gives a 2-approximation if we use the min-sum measure for the two routes.

4 Min-Sum Segment Watchman Routes in Convex Polygons

We examine the min-sum variant of the segment watchman routes problem in convex polygons.

▶ **Lemma 4.1** (⋆). *For convex polygons, each of the two optimal min-sum segment watchman routes visits a consecutive set of convex vertices.*

▶ **Corollary 4.2.** *Problem 2 can be solved in polynomial time in convex polygons.*

5 Conclusion and Future Work

In this abstract, we investigated segment watchman routes in simple polygons. We identified sufficient conditions for two watchman routes to be segment watchman routes, and developed a 2-approximation algorithm for the min-max and the min-sum measure. Furthermore, we argued that the problem of computing min-max segment watchman routes for simple polygons is NP-hard, and concluded that computing min-sum segment watchman routes for convex polygons is possible in polynomial time. We plan to extend the study of Problem 2 to general simple polygons.

The NP-hardness of Problem 1 for three and k watchmen follows easily from an adaption of the proof of Theorem 3.1. We aim to investigate these two problems for $k > 2$ in the future.

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