Two Watchmen's Routes in Staircase Polygons*

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Abstract

We consider the watchman route problem for multiple watchmen in staircase polygons, which are rectilinear x- and y-monotone polygons. For two watchmen, we propose an optimal algorithm that takes quadratic time, improving on the cubic time of the trivial solution. For $m \geq 3$ watchmen, we explain where our approach fails.

1 Introduction

The watchman route problem asks for a shortest route inside a polygon, such that every point in the polygon is visible to some point on the route. It was first introduced by Chin and Ntafos [2], who showed that the problem is NP-hard for polygons with holes, but may be solved efficiently for simple polygons. Given a starting point, an optimal route can be computed in $O(n^3)$ time [8], and finding a solution without a fixed starting point takes a linear factor longer [7].

The Watchman Route Problem has also been considered for multiple watchmen (a problem introduced by Carlsson, Nilsson, and Ntafos [1]). For histograms, efficient algorithms have been proposed for minimizing the total route length (min-sum) [1] and the length of the longest route (min-max) [6]. Here, we are interested in only two watchmen. For this problem, Mitchell and Wynters [4] proved NP-hardness for the min-max objective in simple polygons. Recently, Nilsson and Packer presented a polynomial-time 5.969-approximation algorithm for the same objective in simple polygons [5].

In this paper, we consider a quite restricted class of polygons, staircase polygons, that for two watchmen allows us to assign the responsibility for guarding any edge solely to one of the two watchmen (and seeing all of a polygon's boundary is for two watchmen sufficient to see the polygon). Additionally, we show that the two routes can be separated by a diagonal between two reflex vertices. This enables a polynomial-time algorithm to compute the optimal two watchman routes (for both the min-max and the min-sum objective). Despite staircase polygons being so restricted, some of the observations we make do not hold for three or more watchmen. This indicates a discrepancy in the computational complexity between the watchman route problem for one or two watchmen and for multiple watchmen.

2 Notation and Preliminaries

A polygon is called *rectilinear* (or *orthogonal*) if all its edges are parallel to the x- or the y-axis of a given coordinate system, and *x-monotone* (*y-monotone*) if every line that is

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orthogonal to the x-axis (y-axis) intersects the polygon in exactly one connected interval. A *staircase polygon* is a rectilinear polygon that is both x- and y-monotone. We call the polygonal chains of boundary edges that lie above and below the interior the *ceiling* and the *floor* of the polygon, respectively. We consider the watchman route problem for multiple watchmen in staircase polygons.

▶ Multiple Watchman Route Problem (m-WRP). Given a polygon P, and a number of watchmen m, find a shortest set of m routes, with respect to the min-sum or min-max criterion, such that every point in P is seen from at least one of the routes.

We denote the length of a route w by $\|w\|$, and refer to a solution of the m-WRP as a set of m watchman routes in P. For simplicity, we will refer to a watchman routes also as a watchman. In the following, we consider the m-WRP for the min-sum and the min-max criterion. Any statement on optimal watchman routes holds for either objective, unless stated otherwise.

Let P be a staircase polygon that is not 2-guardable with point guards (as then none of the watchmen would need to walk). As P is x- and y-monotone, we make the following observation:

▶ **Observation 2.1.** *A watchman w with leftmost x-coordinate $x_{\min}$ and rightmost x-coordinate $x_{\max}$ sees at least all points $p \in P$ with $x(p) \in [x_{\min}, x_{\max}]$.*

The analogous statement holds for the bottommost y-coordinate $y_{\min}$ and the topmost y-coordinate $y_{\max}$ of watchman w. Watchman w thus sees the contiguous part of the ceiling between $y_{\min}$ and $x_{\max}$, and the contiguous part of the floor between $x_{\min}$ and $y_{\max}$.

We denote the extensions of edges that are incident to reflex vertices as *cuts*, and identify so-called *essential cuts*. For a single watchman, a simple polygon is seen if all its essential cuts are visited. Clearly, visiting all essential cuts is a necessary condition for a set of watchman routes. A staircase polygon has at most four essential cuts (see Figure 1(a)): the leftmost vertical extension of the floor v_{left}, the lowest horizontal extension of the ceiling h_{bot}, the rightmost vertical extension of the ceiling v_{right}, and the topmost horizontal extension of the floor h_{top}. By "visiting" such an extension, we mean that a watchman route has a point to the left of v_{left}, below h_{bot}, to the right of v_{right}, or above h_{top}. Note that not necessarily all of these four extensions are essential cuts. For the sake of simplicity, we will nevertheless refer to them as such.

For one watchman, an optimal solution is given by the shortest route that visits all four essential cuts. An example is shown in Figure 1(a). By the following theorem proven by Chin and Ntafos [2], such a solution may be computed in linear time.

▶ **Theorem 2.2.** *(Theorem 2 [2], Chin & Ntafos) A shortest watchman route in simple rectilinear polygons can be found in linear time.*

For multiple watchman routes, the watchmen share the responsibility of seeing P. Thus, we aim to find a "good" distribution of responsibilities among the watchmen. For two watchmen, we prove that the polygon may be split into two subpolygons such that an optimal solution to the 2-WRP corresponds to an optimal solution to the WRP in each subpolygon.

3 Computing an Optimal Solution for Two Watchmen

In this section, we investigate the 2-WRP. Let us first state some properties of two optimal watchman routes in staircase polygons. Due to limited space, we omit the proof of Lemma 3.1.

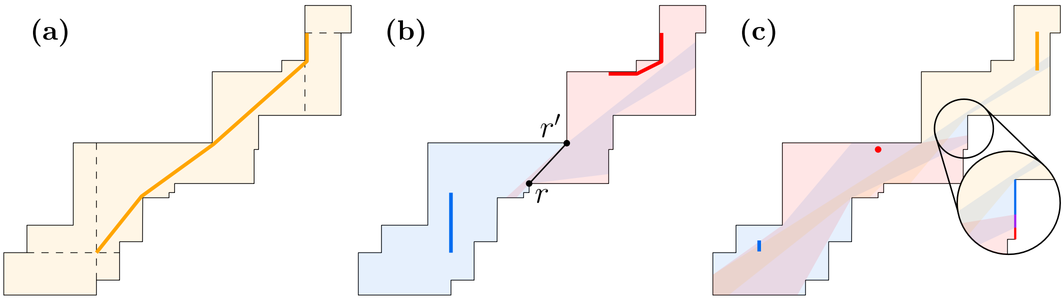

Figure 1 Optimal solutions for (a) one watchman, (b) two watchmen, (c) three watchmen.

▶ **Lemma 3.1.** *Let (w_1^*, w_2^*) be an optimal solution to the 2-WRP in a staircase polygon P. Then, the following properties hold:*

1. *w_1^* and w_2^* do not have any common x- and y-coordinate.*
2. *w_1^* visits the essential cuts h_{bot}, v_{left}, and w_2^* visits the essential cuts h_{top}, v_{right}.*
3. *There exists a pair of reflex vertices (r, r') with r on the floor and r' on the ceiling, such that $\overline{rr'}$ separates w_1^* and w_2^*, see Figure 1(b).*

In the following, we always assume that an optimal solution (w_1^*, w_2^*) obeys Properties 1–3 of Lemma 3.1. In particular, w_1^* lies below and to the left of w_2^*.

▶ **Lemma 3.2.** *In an optimal solution to the 2-WRP in a staircase polygon, for every polygon edge there exists a watchman that sees the edge completely.*

Proof. Let (w_1^*, w_2^*) be an optimal solution, and consider w_1^*. As soon as it crosses the extension of a horizontal floor edge e, it sees e completely since nothing blocks the visibility between w_1^* and e along e's extension. Similarly, w_1^* sees a vertical edge on the ceiling completely as soon as it crosses the edge's extension. Before crossing the extension, w_1^* does not see the respective edge at all. Hence, for any horizontal floor edge (vertical ceiling edge) e, if w_1^* sees any point on e, then it sees all points of e. Similarly, for any horizontal ceiling edge (vertical floor edge) e, if w_2^* sees any point on e, then it sees all points of e. Assume w.l.o.g. that there is a horizontal floor edge e such that no point on e is seen by w_1^*. Then, w_2^* sees e completely as otherwise there are points on e that are not seen by any of w_1^* and w_2^*. ◀

With this, we may split two optimal watchman routes in a particular way.

▶ **Lemma 3.3.** *Let (w_1^*, w_2^*) be an optimal solution in a staircase polygon P. There exists a unique diagonal between a vertex on the floor and a vertex on the ceiling that cuts P into two subpolygons P_1 and P_2 such that w_1^* sees P_1, and w_2^* sees P_2.*

Proof. By Lemma 3.2, every edge is completely seen by a watchman. For a chain of consecutive edges on the floor or ceiling, there cannot be an alteration in the responsibility of the watchmen: Let e_i, e_{i+1}, e_{i+2} be three consecutive edges (on the floor or ceiling). If one watchman sees e_i, e_{i+2} completely, then it also sees e_{i+1}. Hence, there exist vertices on the floor and the ceiling such that w_1^* sees all edges that lie below and to the left of them completely, and w_2^* sees all edges that lie above and to the right of them completely. We call such vertices *breaking points* and show that there exist two breaking points, one on the floor and one on the ceiling, that see each other—these define the unique diagonal. Assume that this is not the case. Let b_f be the lowest-leftmost breaking point on the floor, and b_c be the upper-rightmost breaking point on the ceiling. W.l.o.g., assume that all breaking points on the floor lie to the upper-right of the breaking points on the ceiling (in particular, b_f lies to the upper-right of b_c).

Since b_f and b_c do not see each other, there exist some edges incident to a reflex vertex r that block the visibility. Assume that these edges lie on the ceiling. Then, the horizontal edge incident to r lies above b_c and below b_f, and is seen by w_2^* (by definition of b_c). Hence, w_2^* sees the vertical floor edge v that is hit by the horizontal extension through r (as described in the proof of Lemma 3.2), and thereby also sees the convex vertex on the lower end of v, contradicting the choice of b_f (being the lowest-leftmost breaking point on the floor). ◀

We present an algorithm that finds an optimal split, and thus computes an optimal solution for two watchmen in $O(n^2)$ time. Observe that Lemma 3.2 only holds for two watchmen. For three or more watchmen, some edges may only be seen partially by each watchman in an optimal solution. An example is shown in Figure 1(c), see the magnified part. The blue watchman is in charge of monitoring a part of a vertical floor edge above the red watchman's visibility region. The yellow watchman does not see this edge at all and would have to walk very far to reach the vertical extension of this edge. Therefore, an optimal solution for $m \geq 3$ watchmen may induce a split of the polygon's floor and ceiling into more than m parts each, such that every part is seen by a single watchman. This means that a watchman may be "in charge of" more than one contiguous part of the boundary on the floor and ceiling, respectively.

3.1 A Quadratic-Time Algorithm for Two Watchmen

To compute an optimal solution, because of Lemma 3.3, we consider all diagonals between vertices on the floor and on the ceiling. Any such diagonal splits P into two subpolygons. For each subpolygon, we compute an optimal watchman route using a modified version of the linear-time algorithm proposed by Chin and Ntafos [2], and then combine the two routes to a solution for the 2-WRP in P.

As there are at most quadratically many diagonals to consider, this procedure trivially yields a cubic-time algorithm. However, maintaining a similar structure of the subpolygons by dealing with the diagonals in a certain order allows us to compute many of the watchman routes in amortized constant time.

To this end, we iterate over the vertices on the floor. For each floor vertex p_f, we compute all its diagonals to points on the ceiling, in clockwise order around p_f. If p_f is a convex vertex, then all diagonals have a negative slope. If p_f is a reflex vertex, some diagonals have positive slope. However, we do not need to consider all diagonals with positive slope, but only those two that are followed or preceded by a positive-slope diagonal in the clockwise order. We call those, and the diagonals with negative slopes, *candidate diagonals*; see Figure 3. Every candidate diagonal splits P into two subpolygons, P_1 below and P_2 above the diagonal.

▶ **Lemma 3.4.** *Any diagonal that is not a candidate diagonal induces a solution that is at least as long as the solution induced by some candidate diagonal.*

Proof. First, note that a diagonal of positive slope is spanned between two reflex vertices. Consider w.l.o.g. a non-candidate diagonal $\overline{p_f p_c}$, as seen in Figure 2. Then there is a convex vertex p_c' above p_c that does not yield a diagonal of p_f because $y(p_c) < y(p_f)$. The subpolygon P_2 above $\overline{p_f p_c}$ has the horizontal line through p_c' as an essential cut. Hence, the watchman route in P_2 has points below this cut. There exists a subpolygon induced by a candidate diagonal (incident to p_c and with the other endpoint p_f' below p_f) that also has the horizontal line through p_c' as an essential cut. For this cut, the watchman route in the subpolygon

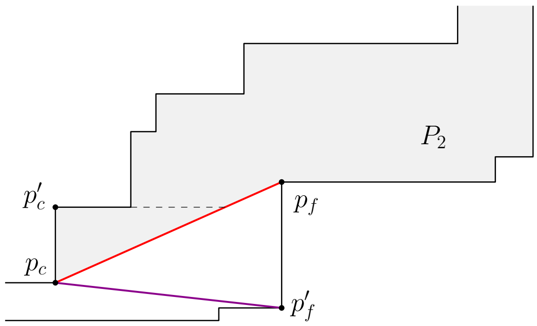

Figure 2 A diagonal with positive slope (red) that is not a candidate: An optimal watchman route in the subpolygon P_2 (marked in gray) needs to visit the same essential cut (dashed line) as an optimal watchman route in the subpolygon induced by $\overline{p_c p_f'}$ (purple).

above the diagonal $\overline{p_c p_f'}$ remains the same, and the watchman route in the subpolygon below is not longer than the one induced by $\overline{p_f p_c}$. ◀

Now we compute a solution for each candidate diagonal in the following manner:

- Step 1: Consider a diagonal with negative slope. Cutting along this diagonal creates only convex vertices in each subpolygon, hence all four essential cuts per subpolygon are rectilinear. The watchman routes touch these extensions, but do not cross them [2]. We compute the optimal solutions for the subpolygons induced by the first diagonal in clockwise order in linear time by Theorem 2.2. In addition, we compute two shortest-path-tree data structures [3]. One is rooted at the first reflex vertex on the floor and stores the shortest paths to all other floor vertices, the other one is rooted at the first reflex vertex on the ceiling and stores the shortest paths to all other ceiling vertices.
  Then, for each diagonal in order, we update the solution in the following way. Moving from one diagonal to the next (i.e., moving from one vertex on the ceiling to the next) alters either the essential cut $v_{\text{right}}(P_1)$ of P_1, or the essential cut $h_{\text{bot}}(P_2)$ of P_2. During this movement, any reflex vertex on the ceiling that was an anchor point can only be released once per vertex p_f, and they are released from right to left. Similarly, any reflex vertex on the floor can be added as an anchor point only once per vertex p_f, and they get added from left to right. Hence, the number of updates per vertex p_f is at most linear.
  When updating the route w_1 in P_1, we move from one vertical extension $v_{\text{right}}(P_1)$ to the next one $v'_{\text{right}}(P_1)$. We use the shortest-path-tree of the floor to check whether vertices on the floor get added to, and the shortest-path-tree of the ceiling to check whether vertices on the ceiling get released from the route. This can be done in amortized constant time [3].
- Step 2: If p_f is a reflex vertex, we need to consider also the two candidate diagonals with positive slope. Here, the subpolygons' essential cuts differ from those of a staircase polygon: There is exactly one non-rectilinear essential cut, namely the extension of the diagonal. We may nevertheless compute an optimal solution, using the algorithm by Chin and Ntafos [2]. This algorithm defines a set of essential cuts, along which the polygon is reflected. Computing the shortest path from one of these essential cuts to its copy yields the shortest watchman route in the original polygon. Since there are at most five essential cuts, we can try all combinations of subsegments of these essential cuts and apply the Chin-and-Ntafos reduction which takes linear time in each of these constant number of cases.

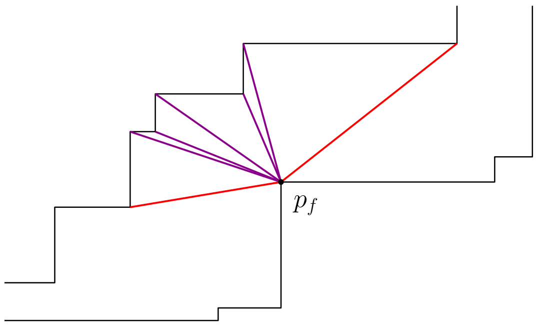

Figure 3 The candidate diagonals of a reflex vertex p_f: there are two candidate diagonals with positive slope (red), and several candidate diagonals with negative slope (purple).

Thus, the computations for each vertex p_f take amortized linear time. As we do this for every vertex on the floor, there are linearly many vertices to consider. With this, we get an optimal solution to the 2-watchman route problem in staircase polygons.

▶ **Theorem 3.5.** *An optimal solution to the* 2*-WRP in staircase polygons can be computed in* $O(n^2)$ *time.*

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