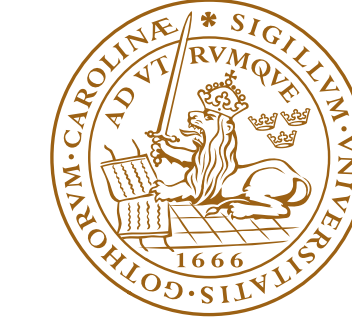


# A characterization of nonexpansive frugal splitting methods

Anton Åkerman, PhD student, Lund University

Dept. of Automatic Control

Supervisors: Pontus Giselsson (LU) and Sebastian Banert (Bremen)



LUND UNIVERSITY

## Motivation & Research Goals

Proximal splitting methods are a simple class of first-order methods suitable high-dimensional, convex and non-smooth optimization problems. We find a simple parameterization and a condition guaranteeing convergence for a certain class of proximal splitting methods. This completely characterizes all methods in this class that are nonexpansive. Joint work with Enis Chenchene and Emanuele Naldi.

## Problem description

We consider methods for solving the optimization problem

$$\text{minimize}_{x \in \mathbb{R}^d} \sum_{i=1}^F f_i(x) + \sum_{i=1}^B g_i(x), \quad (1)$$

for  $B \geq 2$ ,  $F \in \mathbb{N}$  all functions convex, with each  $f_i$  differentiable with  $\beta_i$ -Lipschitz gradient.

Frugal splitting methods use the following building blocks

- One gradient for each differentiable function  $f_i$  per iteration
- One evaluation of the proximal operator, defined as  $\text{prox}_{\gamma g}(x) = \text{argmin}_z \left( g(z) + \frac{1}{2\gamma} \|z - x\|^2 \right)$ , for each function  $g_i$  per iteration
- Linear combinations

Why splitting?

- Proximal operator potentially much cheaper to evaluate for the individual function, functions if prox friendly
- Possible to solve problems distributively

Consider an optimization problem on the form  $\text{minimize}_{x \in C} f(x) = \min_{x \in \mathbb{R}^d} f(x) + \iota_C(x)$ , which can be solved with the proximal gradient method. The iterates of the projected gradient method,  $x_{k+1} = \Pi_C(x_k - \gamma \nabla f(x_k))$ , converge to a minimizer.

## Results

For a splitting method we want to guarantee the following:

- Fixed-point encoding, i.e. algorithm fixed-points always correspond to solutions of our optimization problem.
- Guaranteed convergence (usually by showing the update is so called averaged nonexpansive)

We show that

All frugal splitting methods (that are fixed-point encoding) (1), in which the iterates  $z^k \in \mathbb{R}^{n \times d}$  have minimal dimension  $n = B - 1$ , can be parameterized as

$$\begin{aligned} x_i^k &= \text{prox}_{\gamma_i g_i} \left( \gamma_i L_i x^k + \gamma_i M_i z^k - \gamma_i \sum_{j=1}^F N_{ij} \nabla f_j(G_j x^k) \right) \\ z^{k+1} &= z^k - M^T x^k \end{aligned}$$

for algorithm points

$$\begin{cases} x^k = (x_1^k, x_2^k, \dots, x_B^k)^T \in \mathbb{R}^{B \times d} \\ z^k \in \mathbb{R}^{(B-1) \times d}. \end{cases}$$

and for  $0 < \gamma \in \mathbb{R}^B$ ,  $L \in \mathbb{R}^{B \times B}$  strictly lower triangular, with  $G, N^T \in \mathbb{R}^{F \times B}$  and  $M \in \mathbb{R}^{B \times (B-1)}$ . Additionally we require that  $\mathbb{1}^T \gamma = \mathbb{1}^T L \mathbb{1}$ ,  $M^T \mathbb{1} = 0$  and  $N^T \mathbb{1} = G \mathbb{1} = \mathbb{1}$ .

Our convergence result is as follows.

Let  $\beta \in \mathbb{R}^F$  be the vector of smoothness constants for the functions  $f_i$ . A method on this form is nonexpansive if and only if

$$2 \text{diag}(\gamma)^{-1} - L - L^T - M M^T \succeq (K^T - N) \text{diag}(\beta) (K - N^T) \quad (2)$$

The same results hold also for the more general monotone inclusion problem, with operators either  $\frac{1}{\beta}$ -cocoercive or maximally monotone.

In summary:

- We give a simple, complete characterization of all nonexpansive frugal splitting methods with minimal lifting.
- Condition (2) gives standard parameter bounds for some existing splitting methods.
- Enables simple algorithm design. The choice of sparse matrices simplifies parallelization and use for distributed optimization.

## References

- [1] Frugal Splitting Operators: Representation, Minimal Lifting, and Convergence  
M. Morin, S. Banert, P. Giselsson  
SIAM Journal on Optimization, 2024
- [2] Resolvent splitting for sums of monotone operators with minimal lifting  
Y. Malitsky, M. Tam  
Mathematical Programming, 2023