Rigidity and flexibility using graphs of groups

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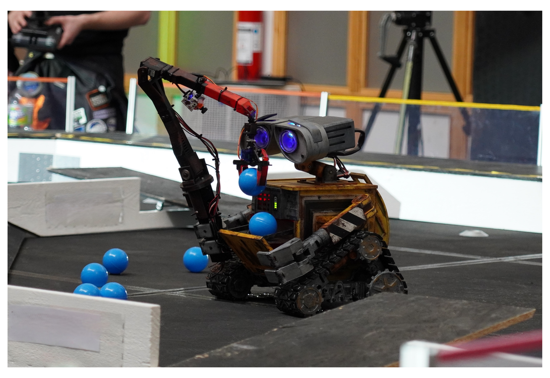
Motivation & Research Goals

The study of motions of articulated structures has many applications in robotics and other fields of science and engineering. Steering robot arms and protein folding are famous examples of problems concerning articulated structures. When the structure contains cycles, these problems are not well understood. We developed a general algebraic framework that allows the study of motions of articulated structures in different geometric settings. The model can be used to build algorithms to determine rigidity of articulated structures.

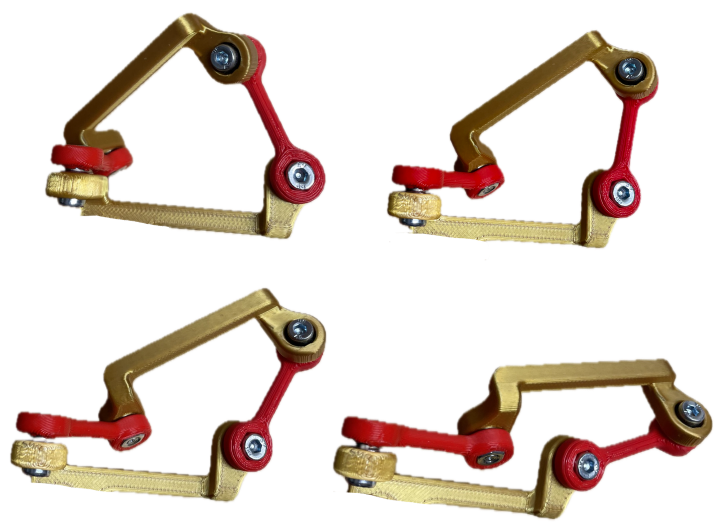
Background



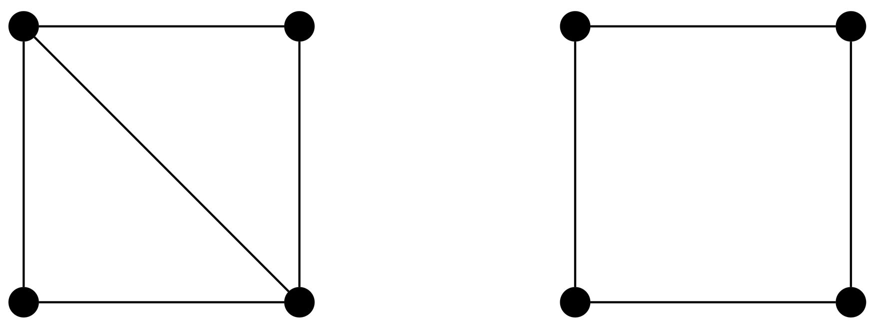
Picture from engineering physics robot competition 2023 at UmU

Understanding how articulated structures can move is of fundamental importance in robotics, engineering and other fields of science. We wanted an algebraic framework to describe motions of such structures.

We have considered applications to **rigidity theory.** Rigidity theory is the mathematical study of **bar-joint frameworks**, which are structures consisting of bars which may rotate around joints.



A bar-joint framework is said to be rigid if it does not admit any motions up to isometries of the underlying space, and otherwise it is called flexible.



Rigid Flexible

For bar-joint frameworks, one has the so-called Maxwell rule. In 2-dimensions this rule says

$$2v - b = m - s,$$

where v is the number of nodes, b is the number of bars m is the dimension of the space of infinitesimal motions (degrees of freedom), and s is the dimension of the space of stresses. This leads to a necessary condition for **generic minimal rigidity** which is also sufficient in dimension 2. Minimal means that taking away any bar results in a non-rigid structure.

Geiringer-Laman theorem: A 2-dimensional bar joint framework is generically minimally rigid if and only if $2v - b = 3$ and for any subset of nodes with v' nodes the amount of bars b' satisfies $b' \leq 2v' - 3$.

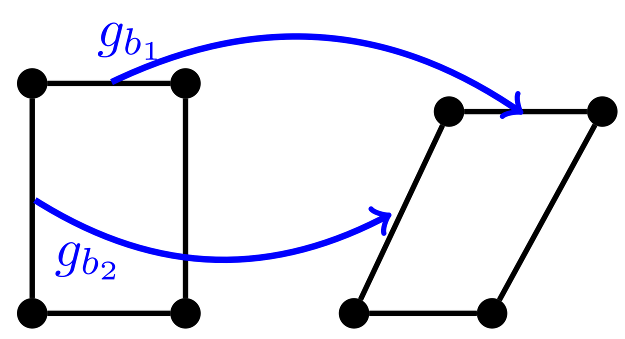
Similar results have been obtained to characterise other problems in combinatorial geometry, though in dimensions 3 and higher the problem of characterising generic rigidity of bar-joint frameworks is still open.

References

[1] *Structural Rigidity and Flexibility Using Graphs of Groups*, K. Stokes, J. Vermant, Arxiv Preprint `https://arxiv.org/abs/2305.07588`

Model

The model we have used is based on group theory and graph theory. By considering how one has to transform bars between two poses of an articulated structure, we find conditions on the transformations.



As in the picture, whenever one is given bars b_1 and b_2 which meet in a point, we have

$$g_{b_1}^{-1} \cdot g_{b_2} \in \mathsf{Stab}(p),$$

where $\mathsf{Stab}(p(v))$ is the **stabiliser** of the point p

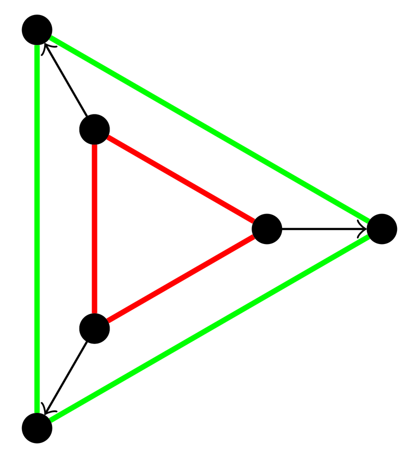
$$\mathsf{Stab}(p) = \{g \in \mathsf{SE}(2) \mid g \cdot p = p\}.$$

This led us to describe the motions in terms of what is called a **graph of groups**. These are fundamental objects in geometric group theory.

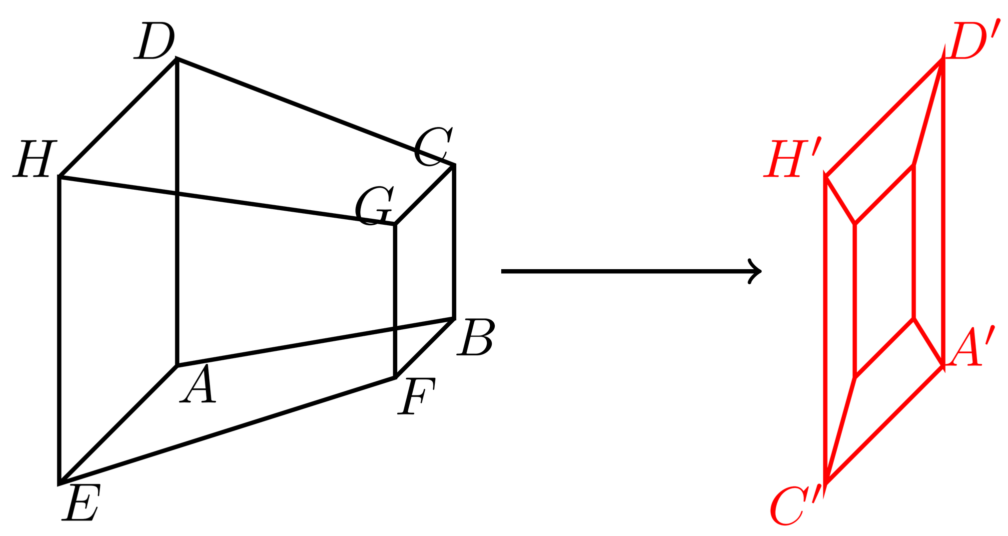
Key results

One can **model various different problems** and **describe relations** using the graph of groups model. Two problems which were known to be dual to each other are pictured below. We described the duality using our model. We can also describe bar-joint frameworks constrained to lie on surfaces using the model.

Parallel redrawings: In how many ways can one draw a figure with prescribed line slopes?



Liftings of a picture: Which 3 dimensional scenes project down to a given picture?



We have also **generalised the Maxwell rule** and **generalised the necessity condition for minimal rigidity** in the Geiringer-Laman theorem to a more general setting.

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