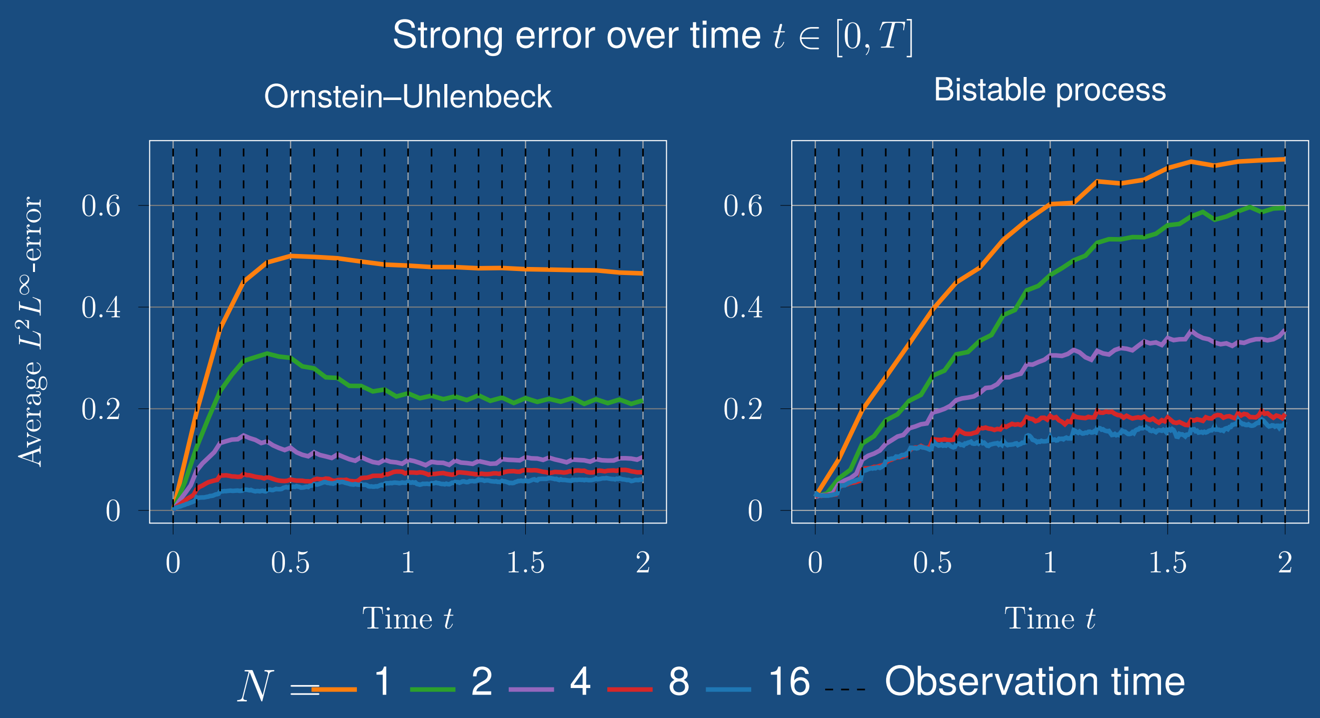
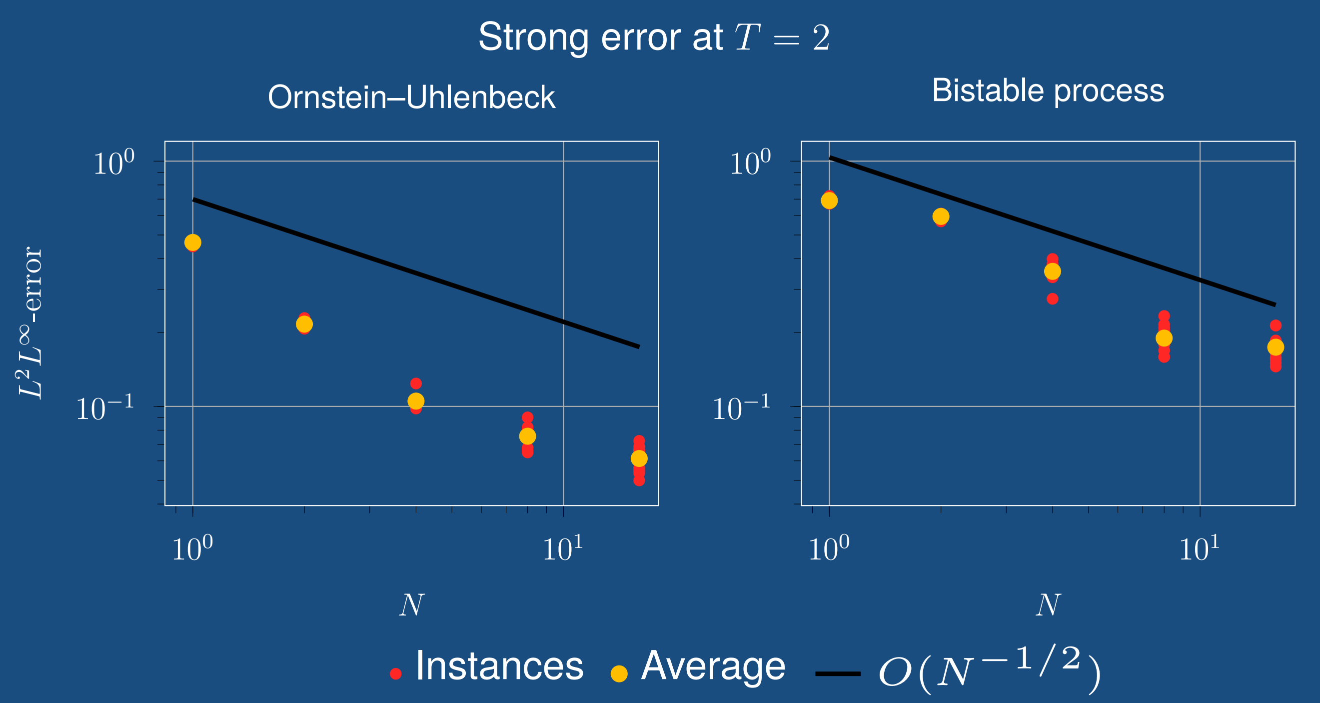
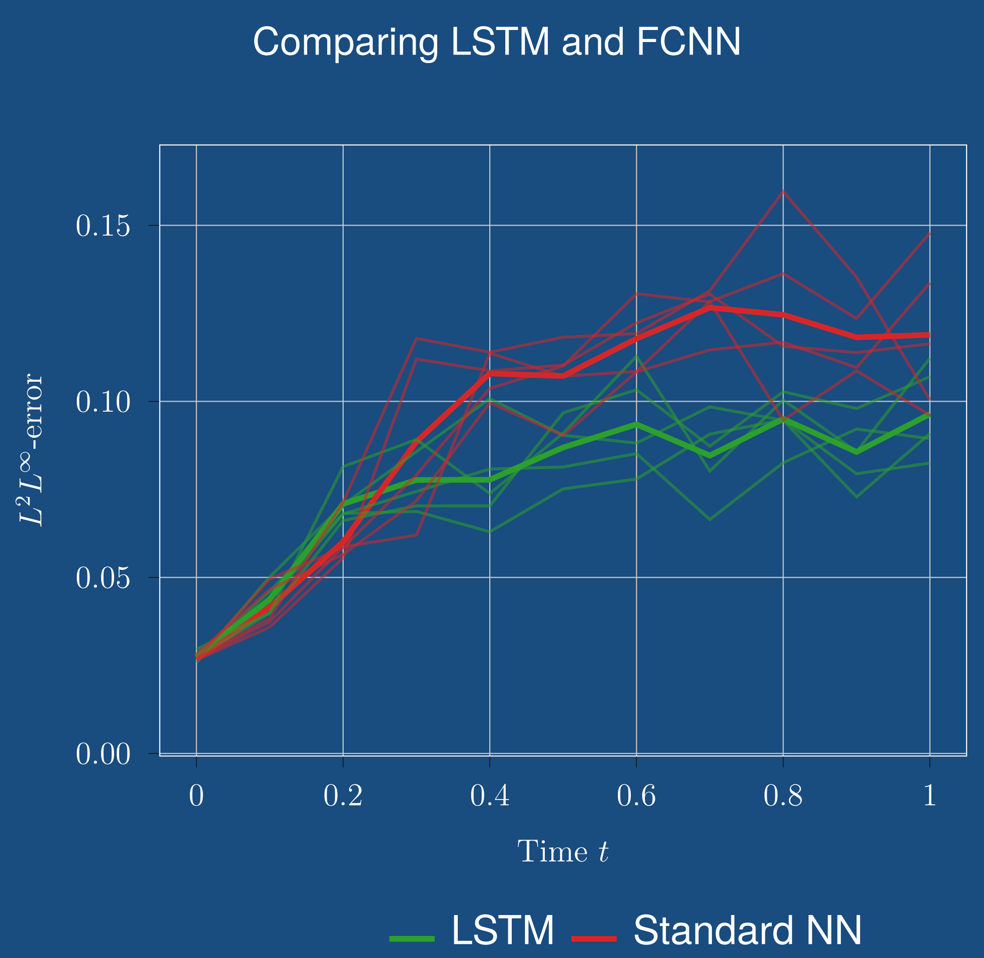
# Strong convergence for a classical prediction-update scheme under a parabolic Hörmander condition based on the Fokker–Planck equation

## Numerical comparison for 5 discretizations over 10 instances:





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# *A convergent scheme for the Bayesian filtering problem*

*Based on the Fokker–Planck equation and deep splitting*

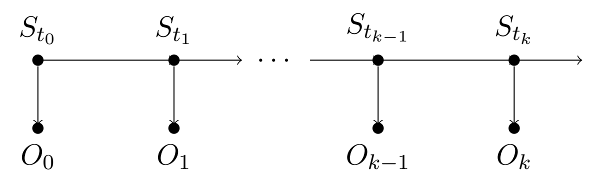
### Bayesian filtering

We assume an underlying state space model

$$\text{(State)}\quad \mathrm{d}S_t = \mu(S_t)\,\mathrm{d}t + \sigma(S_t)\,\mathrm{d}B_t, \quad t \in (0, T]$$

$$S_0 \sim p_0$$

$$\text{(Observations)}\quad O_k \sim \mathcal{N}(h(S_{t_k}), R), \quad k = 0, \dots, K$$




*Main objective*

- Find the probability density $p(S_{t_k} \mid O_{0:k})$ called the **filtering density**

### Problem formulation

The filtering density $p \in C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R})$, associated to $(S, O)$, satisfies the Fokker–Planck equation with discrete measurement updates for $k = 0, \dots, K$ and $t \in (t_k, t_{k+1}]$

$$\text{(Prediction)}\quad p_k(t) = p_k(t_k) + \int_{t_k}^{t} \Big(A p_k(s) + F p_k(s)\Big)\,\mathrm{d}s$$

$$\text{(Update)}\quad p_k(t_k, Y_{0:k}) = p_{k-1}(t_k, Y_{0:k-1}) L(Y_k)$$

where

$$L(Y_k, x) := p(O_k = Y_k | S_{t_k} = x) = \mathcal{N}(Y_k \mid h(x), R)$$

is tractable and $Y$ is incoming data

*Auxiliary objective*

- Approximate the Prediction step by approximating the solution to the Fokker–Planck equation

### Feynman–Kac representation

Introducing the auxiliary process $X$ satisfying

$$X_t = X_0 + \int_0^t \mu(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s, \quad X_0 \sim q_0,$$

we have the Feynman–Kac representation of $p$

$$p_k(t_{k,n+1}, x) = \mathbb{E}\Big[p_k(t_{k,n}, X_{t_N - t_n}^{t_N - t_{n+1}, x}) + \int_{t_N - t_{n+1}}^{t_N - t_n} F p_k(t_{k,N} - s, X_s^{t_N - t_{n+1}, x})\,\mathrm{d}s\Big]$$

### Euler–Maruyama approximations

Apply the (energy-based) **deep splitting** methodology to get the recursive approximation given by

$$\overline{\pi}_{k,n+1} = \operatorname*{arg\,min}_{u \in \{\mathcal{N}_\theta : \theta \in \Theta_k\}} \mathbb{E}\Big|u(X_{T - t_{n+1}}) - (I + F\Delta t)\overline{\pi}_{k,n}(X_{T - t_n})\Big|^2$$

where $\{\mathcal{N}_\theta : \theta \in \Theta_k\} \subset C(\mathbb{R}^d \times \mathbb{R}^{d' \times (k+1)}; \mathbb{R})$

### Strong convergence theorem

Assuming $\mu$ and $\sigma$ satisfy a **parabolic Hörmander condition** there exists $C := C(T, \mu, \sigma, R, K) > 0$, such that

$$\sup_{\substack{k=0,\dots,K \\ n=0,\dots,N}} \big\|p_k(t_{k,n}) - \overline{\pi}_{k,n}\big\|_{L^2(\Omega; L^\infty(\mathbb{R}^d; \mathbb{R}))} \le C N^{-\frac{1}{2}}$$

### Numerical examples

Numerically measuring the strong error with $\sigma(x) = 1$, $p_0 = \mathcal{N}(0, 1)$, $h(x) = x$ and $K = 20$.

**Ornstein–Uhlenbeck:** Let $\mu(x) = -3x$, then the **Kalman filter** provides the exact solution.

**Bistable:** Let $\mu(x) = \frac{2}{5}(5x - x^3)$, then a bootstrap **particle filter** provides an approximate exact solution.