Algorithm Design with Steering Vectors

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Preliminaries

We aim to solve, over a Hilbert space \mathcal{H} , the monotone inclusion problem

Main Problem: Find $\overline{x} \in \mathcal{H}$ such that $0 \in \sum_{i=1}^{m} A_i \overline{x}$.

where $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator. Our tools are the resolvents $J_{\gamma_i A_i} \coloneqq (I + \gamma_i A_i)^{-1}$ for positive stepsizes γ_i (resolvent splitting). This problem generalizes convex problems For simplicity, assume that M = I. After some work, we get that this fundamental inequality holds if and only if

$$\frac{1}{2} \|x^{+} - x\|_{2}^{2} + \langle x^{+} - x, x \rangle + \sum_{i=1}^{m-1} \left(\frac{1}{2} \|y_{i}^{+} - y_{i}\|_{2}^{2} + \langle y_{i}^{+} - y_{i}, y_{i} - \bar{y}_{i} \rangle \right) \\ + \lambda \left\langle \frac{z_{m} - p_{m}}{\gamma_{m}} + \sum_{i=1}^{m-1} \bar{y}_{i}, p_{m} \right\rangle + \sum_{i=1}^{m-1} \lambda \left\langle \frac{z_{i} - p_{i}}{\gamma_{i}} - \bar{y}_{i}, p_{i} \right\rangle$$

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \sum_{i=1}^m f_i(x) \iff \underset{\overline{x} \in \mathbb{R}^n}{\text{Find}} \ 0 \in \sum_{i=1}^m \partial f_i(\overline{x}).$$

We suppose that there exists some solution of primal and dual variables $\bar{\chi} \coloneqq (\bar{x}, \bar{y}_1, \dots, \bar{y}_{m-1}) \in \mathcal{H}^m$ such that $\bar{y}_i \in A_i \bar{x}$ and $\bar{y}_m \coloneqq -\sum_{i=1}^{m-1} \bar{y}_i \in A_m \bar{x}$. The normal workflow of algorithm research:

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Algorithm Design \longrightarrow Lyapunov Analysis.
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But **how** did you come up with the algorithm and **why** does the Lyapunov analysis really work?

Our Workflow: Algorithm Design — Lyapunov Analysis.

We start with arbitrary points $\chi \coloneqq (x, y_1, \dots, y_{m-1}) \in \mathcal{H}^m$ and $\chi^+ \coloneqq (x^+, y_1^+, \dots, y_{m-1}^+) \in \mathcal{H}^m$ and a metric $\|\cdot\|_M$, where $M \in \mathbb{R}^{m \times m}$ is positive definite. Then, the inequality

$$\|\chi - \overline{\chi}\|_M \le \|\chi^+ - \overline{\chi}\|_M \tag{1}$$

will drive the discussion, itself defining χ^+ from χ .

$$-\left\langle x^{+} - x + \lambda \sum_{i=1}^{m} \frac{z_{i} - p_{i}}{\gamma_{i}}, \bar{x} \right\rangle \leq 0$$

which implies that $x^+ \coloneqq x - \lambda \sum_{i=1}^m (z_i - p_i)/\gamma_i$ must hold. After similarly expanding the fundamental quadratic inequality, and using the definition of x^+ , we get that $y_i^+ \coloneqq y_i - \lambda(p_i - p_m)$ must hold.

Algorithm Design

After performing the change of variables

$$\begin{cases} \tilde{z}_i &= z_i - x - \gamma_i y_i, \quad i = 1, \dots, m-1 \\ \tilde{z}_m &= z_m - x + \gamma_m \sum_{i=1}^{m-1} y_i \\ \tilde{p}_i &= p_i - x \end{cases}$$

and using the definitions of x^+ and y_i^+ we get that the fundamental quadratic inequality holds if and only if

The Final Quadratic Inequality:

Lyapunov Analysis

Let $\{z_i\}_{i=1}^m$ be a collection of points in \mathcal{H} and $p_i \coloneqq J_{\gamma_i A_i} z_i$. Using the monotonicity of A_i for $i \in [1, m-1]$ we get that

$$\left\langle \frac{z_i - p_i}{\gamma_i} - \bar{y}_i, p_i - \bar{x} \right\rangle \ge 0$$

and the monotinicity of A_m we get that

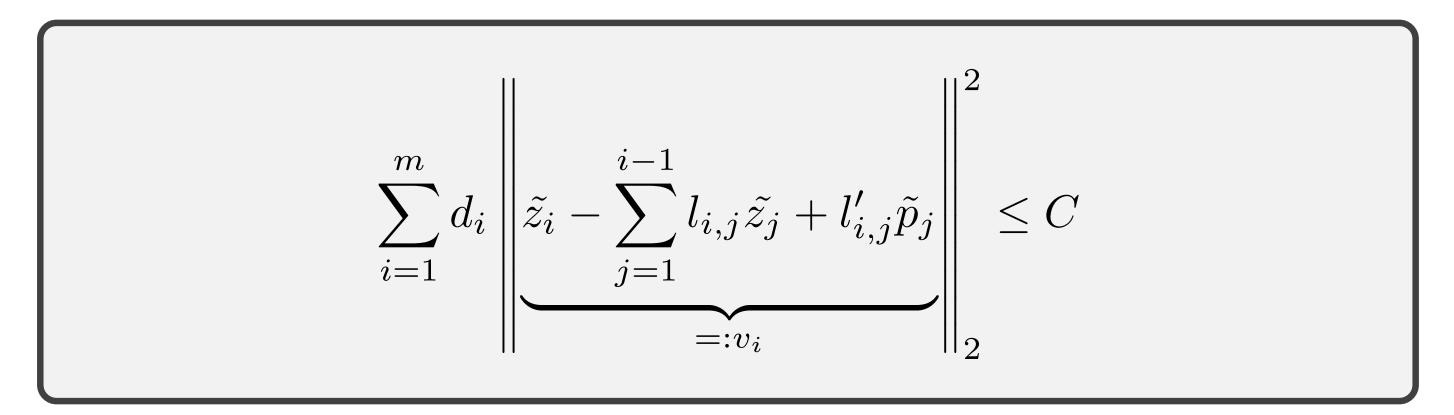
$$\left\langle \frac{z_m - p_m}{\gamma_m} + \sum_{i=1}^{m-1} \bar{y}_i, p_m - \bar{x} \right\rangle \ge 0$$

We get that (1) holds if the following inequality holds.

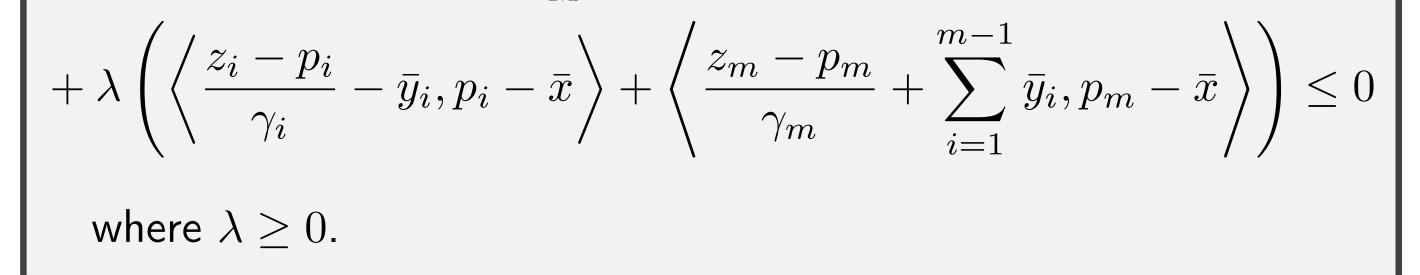
A Fundamental Quadratic Inequality: $\|\chi - \overline{\chi}\|_{M} - \|\chi^{+} - \overline{\chi}\|_{M}$ (/z) = n (-1)

$$\mathbf{Q}(\tilde{\mathbf{z}}, \tilde{\mathbf{p}}, \lambda) \coloneqq \frac{\lambda^2}{2} \left\| \begin{bmatrix} \sum_{i=1}^m (\tilde{z}_i - \tilde{p}_i) / \gamma_i \\ \tilde{p}_1 - \tilde{p}_m \\ \vdots \\ \tilde{p}_{m-1} - \tilde{p}_m \end{bmatrix} \right\|_2^2 + \lambda \sum_{i=1}^m \left\langle \frac{\tilde{z}_i - \tilde{p}_i}{\gamma_i}, \tilde{p}_i \right\rangle \le 0.$$

This inequality we can factorize with the LDL^{\top} factorization, which transforms the problem of *finding convergent algorithms* in \mathcal{H}^m into an *eigenvalue problem* over \mathbb{R}^{2m} . Indeed, we can show that for λ small enough, the signature of this quadratic form equals (m, m, 0) which leads us to



where the constants $d_i, l_{i,j}, l'_{i,j}$ and C are precomputed from the LDL^+ factorization, such that $d_i, C > 0$. The steering vectors v_i can then be



Our first step will be to remove the *dependence* of the solution $\overline{\chi}$ from this inequality.

designed by the user to achieve faster convergence.

Future Work

- Explore which algorithms we capture with this design method.
- Incorporate such that x^+ can depend on a longer history of iterations.
- Generalize this methodology so that it handles cocoercive operators.

