Learning Group Invariant Ricci-flat Metrics

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Abstract

We present invariant machine learning models that approximate the Ricci-flat metrics on Calabi–Yau manifolds with discrete symmetries. By integrating the ϕ -model, in *cymetric* package, with G-invariant layers that project input data to a symmetry group's fundamental domain, the models achieve more accurate metric approximations compared to standard ϕ -models. The method is also applicable to computing Ricci-flat metrics on smooth CY quotients, as demonstrated on a $\mathbb{Z}_5^2$ quotient of a quintic CY manifold.

Calabi-Yau Manifold

A Calabi-Yau (CY) manifold is a compact, complex, Kähler 3-fold with a non-vanishing holomorphic top form Ω. Yau's theorem states that for a CY manifold with metric g, there exists a unique Ricci-flat metric g_{cy} within the same cohomology class. This metric satisfies the **Monge–Ampère (MA)** equation:

$$J_{cy} \wedge J_{cy} \wedge J_{cy} = \kappa\, \Omega \wedge \overline{\Omega}$$

where κ is a real constant and J_{cy} is the associated Kähler 2-form to g_{cy}. The Ricci-flat Kähler form J_{cy} can be written as

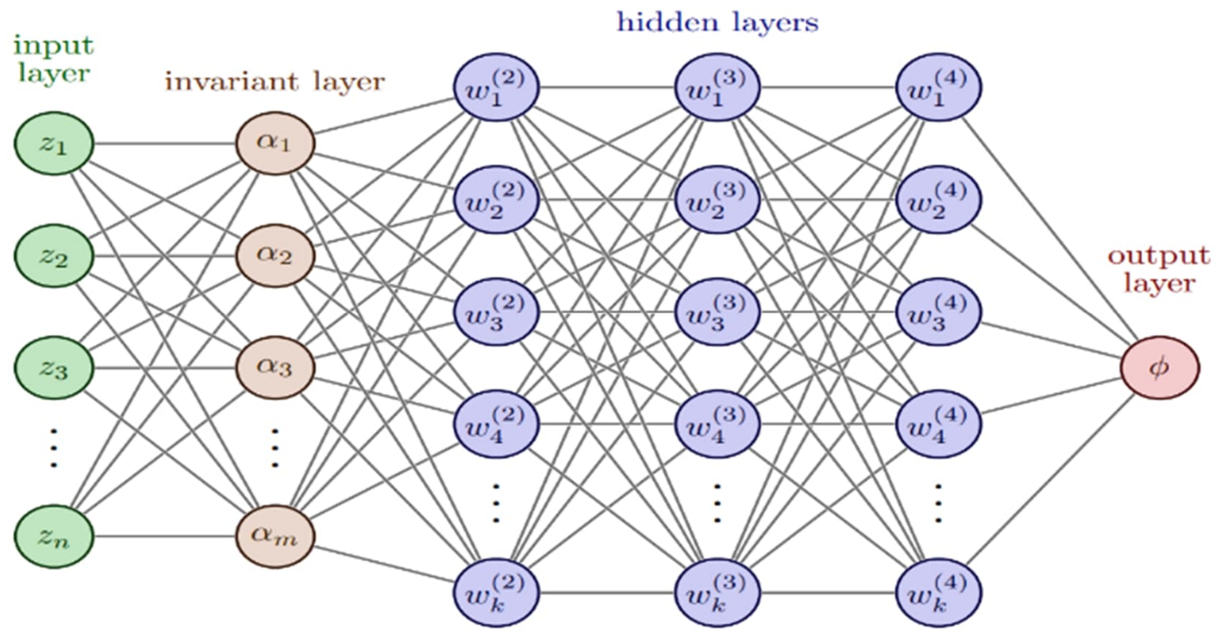
$$J_{cy} = J + \mathrm{i}\partial\overline{\partial}\,\phi,$$

for some reference 2-form J. While there are no analytical solutions for J_{cy} , numerical approximations can be made by approximating the real function ϕ. The ϕ-model in the Tensorflow package *cymetric*[2] accomplishes this using a neural network.

Discrete Symmetries on CY

A complete intersection CY (CICY), X is given as the vanishing locus of a set of homogenous polynomials $\{p_i\}$. The group G of discrete symmetries of $\{p_i\}$ translate into discrete isometries on X w.r.t g_{cy} which translate into invariances for the function ϕ.

To make the ϕ -model G-invariant, we augment it with a ***G*-canonicalization** (invariant) layer.



Canonicalization[1,3]

A fundamental domain F of G acting on X is a subset of X containing a unique representative for each class X/G.

A *G*-canonicalization $h\colon X \to F$ is a map satisfying $[h(x)] = [x]$ for classes in X/G i.e.

$$[h(g.x)] = [g.x] = [x] = [h(x)] \Rightarrow h(g.x) = h(x)$$

Our Contribution

We construct invariant models for different CICYs using canonicalizations, and show that the invariant models obtain lower integrated error in solving the MA equation.

Example: Consider the Fermat quintic CICY in $\mathbb{C}P^4$ given as the vanishing locus of

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0.$$

We have discrete symmetries given by permutations, S_5, and scaling by fifth roots of unity in each homogenous coordinates, $\mathbb{Z}_5^5$. We also have continuous symmetries given by scaling with $\mathbb{C}^*$ since we work with homogenous coordinates.

We build the following canonicalizations for each of these groups of symmetries:

$$h_{\mathbb{C}^*}(z_0, z_1, z_2, z_3, z_4) \coloneqq \frac{1}{z_{max}}(z_0, z_1, z_2, z_3, z_4), \qquad z_{max} \coloneqq \underset{z_i}{\operatorname{argmax}}\,|z_i|$$

$$h_{S_5}(z_0, z_1, z_2, z_3, z_4) \coloneqq (z_{i_0}, z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4}), \qquad |z_{i_j}| \geq |z_{i_{j+1}}|$$

$$h_{\mathbb{Z}_5^5}(z_0, z_1, z_2, z_3, z_4) \coloneqq \left(|z_0| \exp i\left(\arg\,(z_0)\; mod\; \frac{2\pi}{5}\right), \dots, |z_4| \exp i\left(\arg\,(z_4)\; mod\; \frac{2\pi}{5}\right)\right)$$

Model	σ measure
FS metric	0.3720
Dense	0.0106 ± 0.0011
Hom.	0.0100 ± 0.0006
Hom. + Perm.	0.0058 ± 0.0007
Hom. + Perm. + Rootsc.	0.0040 ± 0.0012

References

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